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Effective actions for massive Kaluza-Klein states on $AdS_3 \times S^3 \times S^3$

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Abstract: We construct the effective supergravity actions for the lowest massive Kaluza-Klein states on the supersymmetric background $AdS_3 \times S^3 \times S^3$. In particular, we describe the coupling of the supergravity multiplet to the lowest massive spin-3/2 multiplet $(\frac{1}{2}, \frac{1}{2}; \frac{1}{2}, \frac{1}{2})_{\rm S}$ which contains 256 physical degrees of freedom and includes the moduli of the theory. The effective theory is realized as the broken phase of a particular gauging of the maximal three-dimensional supergravity with gauge group $\mathrm{SO}(4) \times \mathrm{SO}(4)$. Its ground state breaks half of the supersymmetries leading to 8 massive gravitinos acquiring mass in a super Higgs effect. The holographic boundary theory realizes the large $\mathcal{N} = (4, 4)$ superconformal symmetry.

Keywords: AdS-CFT and dS-CFT Correspondence, Extended Supersymmetry, Supersymmetry Breaking, Supergravity Models.

Contents

1. Introduction

Among the celebrated AdS/CFT dualities [1], the correspondence between two-dimensional conformal field theories and string theories on AdS_3 is of particular importance not at least due to the infinite dimensional structure of the conformal group. Originally, the interest was mainly focused on string theory on $AdS_3 \times S^3 \times M_4$, with $M_4 = K3$ or T^4 arising as the near-horizon geometry of the D1-D5 system [2]. The dual conformal field theory is described by a non-linear σ-model whose target space is a deformation of the symmetric product orbifold $Sym^N(M_4)$. More recently, the focus has also turned to the duality involving string theory on $AdS_3 \times S^3 \times S^3 \times S^1$ [3]–[8]. This geometry is half maximally supersymmetric and arises in the near-horizon limit of the so-called double D1-D5 system [9, 4]. Its isometries form two copies of the supergroup $D^1(2, 1; \alpha)$. The ratio of brane charges α here coincides with the ratio of the two sphere radii. Correspondingly, the dual conformal field theory should realize the *large* $\mathcal{N}=4$ superconformal algebra $\mathcal{A}_\gamma$.[1] It has been conjectured to be related to a deformation of the σ-model whose target space is the symmetric product $Sym^N(\mathrm{U}(2))$ [3, 4]. Despite the larger symmetry, this holographic duality is still far less understood than the case of the single D1-D5 system.

In the supergravity limit, the Kaluza-Klein (KK) modes on the $AdS_3 \times S^3 \times M_4$ geometry are dual to chiral primary operators in the conformal field theory. Although CFT calculations have been mainly performed at the orbifold point, where the supergravity approximation breaks down, nontrivial tests of the dualities are possible for quantities protected by non-renormalization theorems; in particular, BPS spectra and elliptic genera were matched successfully [10]. From the supergravity side this essentially requires linearization of the ten-dimensional field equations around the AdS_3 background. Knowledge of the full nonlinear three-dimensional effective supergravity theory on the other hand contains considerable information about the conformal field theory even beyond the level of the spectrum. For instance, the full nonlinear couplings of the supergravity fields encode the higher order correlation functions on the CFT side. Furthermore, in order to analyze renormalization group flows in the field theory via the dual supergravity description, one needs the full scalar potential of the supergravity theory which in particular encodes information about the infrared fixpoints of these flows [11].

For the case of $AdS_3 \times S^3$, the full effective supergravity theory has been constructed in [12], drawing on the special properties of gauged supergravities in three dimensions [13]–[17]. The interactions of the infinite Kaluza-Klein towers of massive spin-1 multiplets with the massless $\mathcal{N}=8$ supergravity multiplet have been described in terms of a gauged supergravity over a single irreducible coset space. It comes with a local SO(4) gauge symmetry related to the isometries of the S^3 sphere — and in the holographic context to the R-symmetry of the conformal field theory. The spin-1 fields acquire mass in a three-dimensional variant of the Brout-Englert-Higgs mechanism involving an infinite number of fields. An open problem has remained the inclusion of the infinite tower of massive spin-2 multiplets in this analysis.

On the $AdS_3 \times S^3 \times S^3$ background, the Kaluza-Klein spectrum organizes into infinite towers of massive spin-3/2 and massive spin-2 multiplets. Massive spin-3/2 fields can be realized through spontaneously broken local supersymmetry and we can therefore be optimistic that at least some part of this Kaluza-Klein tower can be incorporated into a supergravity description. In this paper we will focus on the lowest multiplets appearing in the KK spectrum. In particular, we will construct the effective theory that describes the interactions between the massless supergravity multiplet and the lowest massive spin-3/2 multiplet $(\frac{1}{2},\frac{1}{2};\frac{1}{2},\frac{1}{2})_{\mathrm{S}}$.[2]

[1]In contrast to the small $\mathcal{N}=4$ superconformal algebra realized by the boundary theory of the single D1-D5 system, the large $\mathcal{N}=4$ algebra $\mathcal{A}_\gamma$ contains two affine $\widehat{\mathfrak{su}(2)}$ factors, related to the isometries of the two S^3 spheres.

[2]For the notation of supermultiplets we follow [4] and denote by $(\ell_L^+,\ell_L^-;\ell_R^+,\ell_R^-)_{\mathrm{S}}$ the short supermultiplet generated from the highest weight state $(\ell_L^+,\ell_L^-;\ell_R^+,\ell_R^-)$ where $\ell_{L,R}^\pm$ denote the spins under the various SU(2) factors of SO(4) $\times$ SO(4), see section 2.1 for details.

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This massive multiplet contains 256 physical degrees of freedom, among them eight massive spin-3/2 fields. It is of particular interest for several reasons. First of all, this multiplet is the only one appearing in the Kaluza-Klein spectrum that carries a massless scalar field which is invariant under the full $\mathrm{SO}(4) \times \mathrm{SO}(4)$. These fields correspond to the moduli of the theory, i.e. they are dual to marginal operators of the conformal field theory. Second, the massive spin-3/2 multiplet is the lowest massive multiplet of the particular type $(\ell_L, \ell_L; \ell_R, \ell_R)_{\mathrm{S}}$, which implies that its conformal dimensions are protected throughout the moduli space. In contrast, generic multiplets appearing in the Kaluza-Klein tower are expected to receive quantum corrections [4, 5]. This is a peculiarity of the theories with large $\mathcal{N} = 4$ symmetry, as the BPS condition of the underlying superconformal algebra $\mathcal{A}_\gamma$ is nonlinear. Finally, together with the supergravity multiplet, the multiplet $(\frac{1}{2}, \frac{1}{2}; \frac{1}{2}, \frac{1}{2})_{\mathrm{S}}$ sits in the same short representation of the superconformal algebra $\mathcal{A}_\gamma$. The combined theory may thus be relevant to realize the full $\mathcal{A}_\gamma$ symmetry.

In this paper we will construct the effective three-dimensional supergravity theory describing the interactions of this massive multiplet as a particular gauging of the maximal $\mathcal{N} = 16$ theory in its broken phase. In particular, we give the complete scalar potential of the theory as a function on the coset manifold $E_{8(8)}/\,\mathrm{SO}(16)$ which yields the correct KK scalar, vector and gravitino masses through a supersymmetric Brout-Englert-Higgs mechanism. As a by-product this construction yields a new example of a maximal gauged supergravity theory within the classification initiated in [13, 18]. Interestingly, and in contrast to previously known examples, it does not possess a symmetric phase but its ground state spontaneously breaks half of the supersymmetries down to an $\mathcal{N} = (4, 4)$ supersymmetric AdS geometry; correspondingly eight of the gravitinos acquire mass.

This paper is organized as follows. In section 2 we review the Kaluza-Klein spectrum on $AdS_3 \times S^3 \times S^3$, as it has been computed in [4]. We describe in detail the lowest multiplets appearing in the infinite Kaluza-Klein tower. The effective three-dimensional theories describing the couplings of the lowest massive spin-1/2 and spin-1 multiplets to the massless $\mathcal{N} = 8$ supergravity multiplet can be constructed along the lines of [14, 12] as particular gauged $\mathcal{N} = 8$ supergravities. Particular attention is paid to the massive spin-3/2 multiplet $(\frac{1}{2}, \frac{1}{2}; \frac{1}{2}, \frac{1}{2})_{\mathrm{S}}$. We argue that its field content suggests the description as a particular gauged maximal $\mathcal{N} = 16$ supergravity theory.

Section 3 then presents the detailed construction of this theory. We start with a brief summary of the general three-dimensional maximal gauged supergravity theory in section 3.1. The generic theory may be described as a deformation of the maximal ungauged theory with global $E_{8(8)}$ symmetry. The deformation is completely specified by the choice of the so-called embedding tensor describing the minimal couplings between vector and scalar fields, which in turn is subject to a set of algebraic consistency constraints. We identify the relevant embedding tensor in section 3.3 and verify that it is a solution to all the algebraic constraints. In section 3.4 we prove the existence of a ground state in this theory which spontaneously breaks half of the supersymmetries and exhibits the correct background isometry group $D^1(2, 1; \alpha) \times D^1(2, 1; \alpha)$. We compute the mass spectrum by linearizing the field equations around this ground state and show that it indeed coincides with the prediction from the Kaluza-Klein analysis. As a first application, we compute

h			
h_0		(ℓ^+, ℓ^-)	
$h_0 + \frac{1}{2}$	$(\ell^+ - \frac{1}{2}, \ell^- - \frac{1}{2})$	$(\ell^+ + \frac{1}{2}, \ell^- - \frac{1}{2})$	$(\ell^+ - \frac{1}{2}, \ell^- + \frac{1}{2})$
$h_0 + 1$	$(\ell^+, \ell^- - 1)$	$(\ell^+ - 1, \ell^-)$	(ℓ^+, ℓ^-)
$h_0 + \frac{3}{2}$		$(\ell^+ - \frac{1}{2}, \ell^- - \frac{1}{2})$	

Table 1: The generic short supermultiplet $(\ell^+, \ell^-)_{\rm S}$ of $D^1(2,1;\alpha)$, with $h_0 = \frac{1}{1+\alpha}\,\ell^+ + \frac{\alpha}{1+\alpha}\,\ell^-$.

in section 3.5 the scalar potential for the scalar fields which are singlets under the gauge group. In particular, this confirms the uniqueness of the ground state. In section 4 we outline future directions of work.

2. Kaluza-Klein supergravity on $AdS_3 \times S^3 \times S^3$

2.1 Supergravity spectrum

We start by reviewing the Kaluza-Klein spectrum of ten-dimensional supergravity compactified on $AdS_3 \times S^3 \times S^3 \times S^1$ following [4]. As we will not discuss the higher Kaluza-Klein modes on the S^1, this corresponds to effectively starting from nine-dimensional supergravity. The spectrum can then be derived by purely group-theoretical methods.

The background isometry supergroup under which the spectrum organizes is the direct product of two $\mathcal{N} = 4$ supergroups

$$D^1(2,1;\alpha)_L \times D^1(2,1;\alpha)_R\,, \tag{2.1}$$

in which each factor combines a bosonic $\mathrm{SO}(3) \times \mathrm{SO}(3) \times \mathrm{SL}(2,\mathbb{R})$ with eight real supercharges (see [19] for the exact definitions). More precisely, the noncompact factors $\mathrm{SL}(2,\mathbb{R})_L \times \mathrm{SL}(2,\mathbb{R})_R = \mathrm{SO}(2,2)$ join into the isometry group of AdS_3 while the compact factors

$$G_{\rm c} = \mathrm{SO}(3)^+_L \times \mathrm{SO}(3)^-_L \times \mathrm{SO}(3)^+_R \times \mathrm{SO}(3)^-_R\,, \tag{2.2}$$

build the isometry groups $\mathrm{SO}(4)^\pm \equiv \mathrm{SO}(3)^\pm_L \times \mathrm{SO}(3)^\pm_R$ of the two spheres $S^{3+} \times S^{3-}$. Accordingly, this group will show up as the gauge group of the effectively three-dimensional supergravity action. The parameter α of (2.1) describes the ratio of the radii of the two spheres S^3.

The spectrum of the three-dimensional supergravity theory combines into supermultiplets of the group (2.1). A short $D^1(2,1;\alpha)$ supermultiplet is defined by its highest weight state $(\ell^+, \ell^-)^{h_0}$, where $\ell^\pm$ label spins of $\mathrm{SO}(3)^\pm$ and $h_0 = \frac{1}{1+\alpha}\,\ell^+ + \frac{\alpha}{1+\alpha}\,\ell^-$ is the charge under the Cartan subgroup $\mathrm{SO}(1,1) \subset SL(2,\mathbb{R})$. The corresponding supermultiplet which we will denote by $(\ell^+, \ell^-)_{\rm S}$ is generated from the highest weight state by the action of three out of the four supercharges $G^a_{-1/2}$ $(a = 1, ..., 4)$ and carries $8(\ell_+ + \ell_- + 4\ell_+\ell_-)$ degrees of freedom. Its $\mathrm{SO}(3)^\pm$ representation content is summarized in table 1.

h	(ℓ^+,ℓ^-)
$\frac{\alpha}{2(1+\alpha)}$	$(0,\frac{1}{2})$
$\frac{2\alpha+1}{2(1+\alpha)}$	$(\frac{1}{2},0)$

h	(ℓ^+,ℓ^-)
$\frac{\alpha}{1+\alpha}$	$(0,1)$
$\frac{3\alpha+1}{2(1+\alpha)}$	$(\frac{1}{2},\frac{1}{2})$
$\frac{2\alpha+1}{1+\alpha}$	$(0,0)$

h	(ℓ^+,ℓ^-)
$\frac{1}{2}$	$(\frac{1}{2},\frac{1}{2})$
1	$(0,0)+(0,1)+(1,0)$
$\frac{3}{2}$	$(\frac{1}{2},\frac{1}{2})$
2	$(0,0)$

Table 2: The lowest short supermultiplets $(0,\frac{1}{2})_{\rm S}$, $(0,1)_{\rm S}$, and $(\frac{1}{2},\frac{1}{2})_{\rm S}$ of $D^1(2,1;\alpha)$.

The generic long multiplet $(\ell^+,\ell^-)_{\rm long}$ instead is built from the action of all four supercharges $G^a_{-1/2}$ on the highest weight state and carries $16\,(2\ell_++1)(2\ell_-+1)$ degrees of freedom. Its highest weight state satisfies the unitarity bound $h\geq\frac{1}{1+\alpha}\,\ell^+ + \frac{\alpha}{1+\alpha}\,\ell^-$. In case this bound is saturated, the long multiplet decomposes into two short multiplets (table 1) according to

$$(\ell^+,\ell^-)_{\rm long} = (\ell^+,\ell^-)_{\rm S}\oplus\left(\ell^+ +\frac{1}{2},\ell^- +\frac{1}{2}\right)_{\rm S}\,. \tag{2.3}$$

The lowest short supermultiplets $(0,\frac{1}{2})_{\rm S}$, $(0,1)_{\rm S}$, and $(\frac{1}{2},\frac{1}{2})_{\rm S}$ of $D^1(2,1;\alpha)$ are further degenerate and collected in table 1, and similar for $\ell^+\leftrightarrow\ell^-$, $\alpha\leftrightarrow 1/\alpha$.

Short representations of the full supergroup (2.1) are constructed as tensor products of the supermultiplets (table 1), and correspondingly will be denoted by $(\ell^+_L,\ell^-_L;\ell^+_R,\ell^-_R)_{\rm S}$. The quantum numbers which denote the representations of the AdS_3 group SO(2,2) are labeled by numbers s and Δ, which encode the AdS analogue of spin and mass, respectively. They are related to the values of h_R and h_L by $s=h_R-h_L, \Delta=h_L+h_R$.

The massive Kaluza-Klein spectrum of nine-dimensional supergravity on the $AdS_3\times S^3\times S^3$ background has been computed in [4]. It can be summarized in supermultiplets as

$$\begin{aligned}&\bigoplus_{\ell^+\geq 0,\ell^-\geq 1/2}(\ell^+,\ell^-;\ell^+,\ell^-)_{\rm S}\;\oplus\bigoplus_{\ell^+\geq 1/2,\ell^-\geq 0}(\ell^+,\ell^-;\ell^+,\ell^-)_{\rm S}\oplus\\ &\oplus\bigoplus_{\ell^+,\ell^-\geq 0}\left(\left(\ell^+,\ell^-;\ell^+ +\frac{1}{2},\ell^- +\frac{1}{2}\right)_{\rm S}\oplus\left(\ell^+ +\frac{1}{2},\ell^- +\frac{1}{2};\ell^+,\ell^-\right)_{\rm S}\right).\end{aligned} \tag{2.4}$$

Note that the multiplets $(\ell^+,\ell^-;\ell^+,\ell^-)_{\rm S}$ generically contain massive fields with spin running from 0 to $\frac{3}{2}$, whereas multiplets of the type $(\ell^+,\ell^-;\ell^+ +\frac{1}{2},\ell^- +\frac{1}{2})_{\rm S}$ represent massive spin-2 multiplets.

In addition, there is the massless supergravity multiplet $(\frac{1}{2},\frac{1}{2};0,0)_{\rm S}\oplus(0,0;\frac{1}{2},\frac{1}{2})_{\rm S}$, which consists of the vielbein, eight gravitinos, transforming as

$$\psi^I_\mu:\;\;\left(\frac{1}{2},\frac{1}{2};0,0\right)\oplus\left(0,0;\frac{1}{2},\frac{1}{2}\right), \tag{2.5}$$

under (2.2), and topological gauge vectors, corresponding to the $\mathrm{SO}(4)_L\times\mathrm{SO}(4)_R$ gauge group. As a general feature of three-dimensional supergravity and in accordance with the counting of table 1 it does not contain any physical degrees of freedom.

We should emphasize that except for this supergravity multiplet and one of the lowest massive spin-3/2 multiplets $(\frac{1}{2},\frac{1}{2};\frac{1}{2},\frac{1}{2})_{\rm S}$, all short multiplets appearing in the Kaluza-Klein spectrum (2.4) may combine into long multiplets (2.3) [4]. The conformal weight

$h_L \backslash h_R$	$\frac{\alpha}{2(1+\alpha)}$	$\frac{1+2\alpha}{2(1+\alpha)}$
$\frac{\alpha}{2(1+\alpha)}$	$(0,\frac{1}{2};0,\frac{1}{2})$	$(0,\frac{1}{2};\frac{1}{2},0)$
$\frac{1+2\alpha}{2(1+\alpha)}$	$(\frac{1}{2},0;0,\frac{1}{2})$	$(\frac{1}{2},0;\frac{1}{2},0)$

$h_L \backslash h_R$	$\frac{\alpha}{1+\alpha}$	$\frac{3\alpha+1}{2(1+\alpha)}$	$\frac{2\alpha+1}{1+\alpha}$
$\frac{\alpha}{1+\alpha}$	$(0,1;0,1)$	$(0,1;\frac{1}{2},\frac{1}{2})$	$(0,1;0,0)$
$\frac{3\alpha+1}{2(1+\alpha)}$	$(\frac{1}{2},\frac{1}{2};0,1)$	$(\frac{1}{2},\frac{1}{2};\frac{1}{2},\frac{1}{2})$	$(\frac{1}{2},\frac{1}{2};0,0)$
$\frac{2\alpha+1}{1+\alpha}$	$(0,0;0,1)$	$(0,0;\frac{1}{2},\frac{1}{2})$	$(0,0,0,0)$

Table 3: The spin-$\frac{1}{2}$ multiplet $(0,\frac{1}{2};0,\frac{1}{2})_{\rm S}$, and the massive spin-1 multiplet $(0,1;0,1)_{\rm S}$.

of these long representations is not protected by anything and may vary throughout the moduli space. This is in contrast to the BPS supergravity spectra usually appearing in Kaluza-Klein sphere compactifications. It gives a distinguished role to the supermultiplet $(\frac{1}{2},\frac{1}{2};\frac{1}{2},\frac{1}{2})_{\rm S}$ that we shall analyze in this paper.

2.2 The lowest supermultiplets and their effective theories

In this paper, we will construct the three-dimensional effective actions that describe the couplings of the lowest multiplets of the full Kaluza-Klein spectrum (2.4). As the $AdS_3 \times S^3 \times S^3$ background preserves half of all the supersymmetries, i.e. 16 real supercharges, the effective three-dimensional theory should be (at least) half maximally supersymmetric. In $D=3$ language this is an $\mathcal{N}=8$ supergravity.

The lowest supermultiplet in the spectrum is the massless supergravity multiplet $(0,0;\frac{1}{2},\frac{1}{2})_{\rm S} \oplus (\frac{1}{2},\frac{1}{2};0,0)_{\rm S}$. As noted above, it does not contain propagating degrees of freedom. It can be effectively described by the difference of two Chern-Simons theories [20, 21]

$$\mathcal{L} = \mathrm{Tr}\left(\mathcal{A}_L \wedge d\mathcal{A}_L + \frac{2}{3}\mathcal{A}_L \wedge \mathcal{A}_L \wedge \mathcal{A}_L\right) - \mathrm{Tr}\left(\mathcal{A}_R \wedge d\mathcal{A}_R + \frac{2}{3}\mathcal{A}_R \wedge \mathcal{A}_R \wedge \mathcal{A}_R\right), \qquad (2.6)$$

with connections $\mathcal{A}_L$, $\mathcal{A}_R$ living on the two factors of the supergroup (2.1).

The lowest massive multiplets in the Kaluza-Klein tower (2.4) are the degenerate multiplets $(0,\frac{1}{2};0,\frac{1}{2})_{\rm S}$ and $(0,1;0,1)_{\rm S}$ (together with $(\frac{1}{2},0;\frac{1}{2},0)_{\rm S}$ and $(1,0;1,0)_{\rm S}$), to which we will refer as the spin-$\frac{1}{2}$ and spin-1 multiplet, respectively, in accordance with their states of maximal spin. Their precise representation content is collected in table 3. Coupling of these multiplets requires to extend the topological lagrangian (2.6) to propagating matter. This is achieved by three-dimensional gauged supergravities [13, 14]. These theories are obtained as deformations of the ungauged $\mathcal{N}=8$ and $\mathcal{N}=16$ theories constructed in [22] which couple supergravity to scalar fields parametrizing the coset spaces $\mathrm{SO}(8,n)/(\mathrm{SO}(8)\times\mathrm{SO}(n))$ and $E_{8(8)}/\,\mathrm{SO}(16)$, respectively.

Specifically, the scalar sector of these theories is given by a gauged coset space σ-model

$$\mathcal{L}_{\rm matter} = e\,\mathrm{Tr}\left\langle [\mathcal{V}^{-1}D_\mu\mathcal{V}]_{\mathfrak{k}}\,[\mathcal{V}^{-1}D^\mu\mathcal{V}]_{\mathfrak{k}}\right\rangle + e\,V(\mathcal{V}) + \text{fermions} \qquad (2.7)$$

on the target space $G/H = \mathrm{SO}(8,n)/(\mathrm{SO}(8)\times\mathrm{SO}(n))$ and $G/H = E_{8(8)}/\,\mathrm{SO}(16)$, respectively. Here, the scalar fields parametrize the G-valued matrix $\mathcal{V}$, $[\cdot]_{\mathfrak{k}}$ denotes the projection of the associated Lie algebra $\mathfrak{g}$ onto its noncompact part, and the covariant derivative is given by

$$D_\mu\mathcal{V} = \partial_\mu\mathcal{V} + gA_\mu^{\mathcal{M}}\,\Theta_{\mathcal{M}\mathcal{N}}\,t^{\mathcal{N}}\,\mathcal{V}\,, \qquad (2.8)$$

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with gauge coupling constant g and a constant symmetric matrix $\Theta_{\mathcal{MN}}$ (the embedding tensor) describing the minimal couplings between gauge fields $A_\mu^{\mathcal{M}}$ and symmetry generators $t^{\mathcal{N}}$ of $\mathfrak{g}$. The full effective theory is completely determined by the choice of this embedding tensor $\Theta_{\mathcal{MN}}$. In particular, the scalar potential $V(\mathcal{V})$ in (2.7) is a unique function of the scalar fields $\mathcal{V}$ and the embedding tensor $\Theta_{\mathcal{MN}}$, see [13, 14] and (3.6) below for details. The topological part (2.6) of the lagrangian takes the explicit form

$$\mathcal{L}_{\rm CS} = -\frac{1}{4}\, e\, R + \frac{1}{4}\, g\, \varepsilon^{\mu\nu\rho}\, A_\mu^{\mathcal{M}}\, \Theta_{\mathcal{MN}} \left(\partial_\nu A_\rho^{\mathcal{N}} - \frac{1}{3} g\, \Theta_{\mathcal{KS}}\, f^{\mathcal{NS}}{}_{\mathcal{L}}\, A_\nu^{\mathcal{K}} A_\rho^{\mathcal{L}} \right) + \text{fermions}\,, \tag{2.9}$$

with structure constants $f^{\mathcal{NS}}{}_{\mathcal{L}}$ of the algebra $\mathfrak{g}$. Likewise, the full fermionic lagrangian is uniquely fixed in terms of the embedding tensor $\Theta_{\mathcal{MN}}$. The gauge group G_0 of the effective theory is identified as the subgroup of G which is spanned by the generators $\{X_{\mathcal{M}} = \Theta_{\mathcal{MN}}\, t^{\mathcal{N}}\}$. At $\Theta_{\mathcal{MN}} = 0$, this theory reduces to the ungauged theory of [22]. In order to be compatible with supersymmetry, the embedding tensor needs to satisfy a number of algebraic constraints which we will describe in more detail below. In turn, any solution to these constraints defines a consistent supersymmetric theory.

The task of constructing the effective supergravity action proceeds in several steps. First, one has to determine the ungauged theory from which the construction starts. For an $\mathcal{N} = 8$ supersymmetric theory this is essentially done by comparing the bosonic and fermionic degrees of freedom to the $16n$ degrees of freedom described by the theory with scalar target space $G/H = \mathrm{SO}(8,n)/(\mathrm{SO}(8) \times \mathrm{SO}(n))$. More specifically, under $H = \mathrm{SO}(8) \times \mathrm{SO}(n)$ the spectrum of this $\mathcal{N} = 8$ supergravity transforms as $(\mathbf{8}_V \oplus \mathbf{8}_C, \mathbf{n})$, where $\mathbf{8}_V$ and $\mathbf{8}_C$ denote the vector and conjugate spinor representation of SO(8), respectively, and $\mathbf{n}$ the vector representation of SO(n). For the maximal $\mathcal{N} = 16$ theory the spectrum transforms as $\mathbf{128}_S \oplus \mathbf{128}_C$ under $H = \mathrm{SO}(16)$. The compact part $G_{\rm c}$ of the desired gauge group G_0 must then be embedded into H such that the spectrum decomposes into the desired spectrum of the effective theory. Finally, the corresponding embedding tensor $\Theta_{\mathcal{MN}}$ must be determined such that i) it projects the group G onto the desired gauge group G_0, and ii) it is compatible with the algebraic constraints imposed by supersymmetry onto this tensor.

Let us illustrate this with the simplest examples. The two spin-$\frac{1}{2}$ multiplets of table 3 each contain 16 degrees of freedom. This suggests that together they are effectively described by a gauging of the $\mathcal{N} = 8$ theory with target space $\mathrm{SO}(8,2)/(\mathrm{SO}(8) \times \mathrm{SO}(2))$. Indeed, one verifies that the field content of $(0, \frac{1}{2}; 0, \frac{1}{2})_{\rm S}$ (table 3) can be lifted from a representation of the gauge group (2.2) to an $\mathbf{8}_V \oplus \mathbf{8}_C$ of SO(8) with the embedding

$$\mathbf{8}_V \to \left(0, \frac{1}{2}; 0, \frac{1}{2}\right) \oplus \left(\frac{1}{2}, 0; \frac{1}{2}, 0\right), \qquad \mathbf{8}_C \to \left(0, \frac{1}{2}; \frac{1}{2}, 0\right) \oplus \left(\frac{1}{2}, 0; 0, \frac{1}{2}\right), \tag{2.10}$$

while the supercharges (2.5) lift to the spinor representation $\mathbf{8}_S$ of SO(8). This corresponds to the canonical embedding $\mathrm{SO}(8) \supset \mathrm{SO}(4) \times \mathrm{SO}(4)$. Hence, the two spin-1/2 multiplets reproduce the field content $(\mathbf{8}_V \oplus \mathbf{8}_C, \mathbf{2})$ of the ungauged $\mathrm{SO}(8,2)/(\mathrm{SO}(8) \times \mathrm{SO}(2))$ theory. It remains to verify that the embedding (2.10) of the gauge group into SO(8,2) is compatible with the constraints imposed by supersymmetry on the embedding tensor $\Theta_{\mathcal{MN}}$.

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As it turns out, these requirements determine $\Theta_{\mathcal{MN}}$ completely up to a free parameter corresponding to the ratio α of the two sphere radii [14]. The effective theory is then completely determined. Its scalar potential has been further investigated in [11] and indeed reproduces the correct scalar masses predicted by table 3.

The coupling of the spin-1 multiplets $(0,1;0,1)_{\rm S} \oplus (1,0;1,0)_{\rm S}$ is slightly more involved due to the presence of massive vector fields but can be achieved by a straightforward generalization of the case of a single S^3 compactification [15, 12]. Here, the effective theory for 128 degrees of freedom is a gauging of the $\mathcal{N} = 8$ theory with coset space $\mathrm{SO}(8,8)/(\mathrm{SO}(8) \times \mathrm{SO}(8))$. The first thing to verify in this case is that the field content of $(0,1;0,1)_{\rm S} \oplus (1,0;1,0)_{\rm S}$ (table 3) can be lifted from a representation of the gauge group (2.2) to an $(\mathbf{8}_V \oplus \mathbf{8}_C, \mathbf{8}_V)$ of $\mathrm{SO}(8) \times \mathrm{SO}(8)$ via the embedding

$$
\begin{aligned}
(\mathbf{8}_V, \mathbf{1}) &\rightarrow \left(0, \frac{1}{2}; 0, \frac{1}{2}\right) \oplus \left(\frac{1}{2}, 0; \frac{1}{2}, 0\right), \quad (\mathbf{8}_C, \mathbf{1}) \rightarrow \left(0, \frac{1}{2}; \frac{1}{2}, 0\right) \oplus \left(\frac{1}{2}, 0; 0, \frac{1}{2}\right), \\
(\mathbf{1}, \mathbf{8}_V) &\rightarrow \left(0, \frac{1}{2}; 0, \frac{1}{2}\right) \oplus \left(\frac{1}{2}, 0; \frac{1}{2}, 0\right), \quad (\mathbf{1}, \mathbf{8}_C) \rightarrow \left(0, \frac{1}{2}; \frac{1}{2}, 0\right) \oplus \left(\frac{1}{2}, 0; 0, \frac{1}{2}\right).
\end{aligned} \tag{2.11}
$$

This corresponds to the embedding of groups $\mathrm{SO}(8) \times \mathrm{SO}(8) \supset \mathrm{SO}(8)_D \supset \mathrm{SO}(4) \times \mathrm{SO}(4)$, where $\mathrm{SO}(8)_D$ denotes the diagonal subgroup of the two SO(8) factors. For instance, (2.11) implies that the bosonic part decomposes as

$$
\begin{aligned}
(\mathbf{8}_V, \mathbf{8}_V) \rightarrow &\left(\left(0, \frac{1}{2}; 0, \frac{1}{2}\right) \oplus \left(\frac{1}{2}, 0; \frac{1}{2}, 0\right)\right) \otimes \left(\left(0, \frac{1}{2}; 0, \frac{1}{2}\right) \oplus \left(\frac{1}{2}, 0; \frac{1}{2}, 0\right)\right) \\
= &\ (0,1;0,1) \oplus (0,1;0,0) \oplus (0,0;0,1) \oplus (1,0;1,0) \oplus (1,0;0,0) \oplus (0,0;1,0) \oplus \\
&\oplus 2 \cdot (0,0;0,0) \oplus 2 \cdot \left(\frac{1}{2}, \frac{1}{2}; \frac{1}{2}, \frac{1}{2}\right),
\end{aligned}
$$

in agreement with table 3 and its conjugate. It is important to note that the massive spin-1 fields show up in this decomposition through their Goldstone scalars.

In order to reproduce the correct coupling for these massive vector fields, the total gauge group $G_0 \subset \mathrm{SO}(8,8)$ is not just the compact factor $G_{\rm c}$ (2.2), but rather takes the form of a semi-direct product

$$
G_0 = G_{\rm c} \ltimes T_{12}\,, \tag{2.12}
$$

with the abelian 12-dimensional translation group T_{12} transforming in the adjoint representation of $G_{\rm c}$ [15]. In the AdS_3 vacuum, these translational symmetries are broken and the corresponding vector fields gain their masses in the corresponding Brout-Englert-Higgs mechanism. The proper embedding of (2.12) into $\mathrm{SO}(8,8)$ is again uniquely fixed by the constraints imposed by supersymmetry on the embedding tensor $\Theta_{\mathcal{MN}}$ up to the free parameter α [12].

Finally, it is straightforward to construct the effective theory that couples both the spin-1/2 and the spin-1 supermultiplets as a gauging of the theory with coset space $\mathrm{SO}(8,10)/(\mathrm{SO}(8) \times \mathrm{SO}(10))$ which obviously embeds the two target spaces described above.

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$h_L \backslash h_R$	$\frac{1}{2}$	1	$\frac{3}{2}$	2
$\frac{1}{2}$	$(\frac{1}{2},\frac{1}{2};\frac{1}{2},\frac{1}{2})$	$(\frac{1}{2},\frac{1}{2};0,0)$ $(\frac{1}{2},\frac{1}{2};0,1)$ $(\frac{1}{2},\frac{1}{2};1,0)$	$(\frac{1}{2},\frac{1}{2};\frac{1}{2},\frac{1}{2})$	$(\frac{1}{2},\frac{1}{2};0,0)$
1	$(1,0;\frac{1}{2},\frac{1}{2})$ $(0,0;\frac{1}{2},\frac{1}{2})$ $(0,1;\frac{1}{2},\frac{1}{2})$	$(0,0;0,0)$ $(0,1;0,0),(0,0;0,1)$ $(1,0;0,0),(0,0;1,0)$ $(0,1;0,1),(1,0;1,0)$ $(0,1;1,0),(1,0;0,1)$	$(0,1;\frac{1}{2},\frac{1}{2})$ $(0,0;\frac{1}{2},\frac{1}{2})$ $(1,0;\frac{1}{2},\frac{1}{2})$	$(0,0;0,0)$ $(0,1;0,0)$ $(1,0;0,0)$
$\frac{3}{2}$	$(\frac{1}{2},\frac{1}{2};\frac{1}{2},\frac{1}{2})$	$(\frac{1}{2},\frac{1}{2};0,0)$ $(\frac{1}{2},\frac{1}{2};0,1)$ $(\frac{1}{2},\frac{1}{2};1,0)$	$(\frac{1}{2},\frac{1}{2};\frac{1}{2},\frac{1}{2})$	$(\frac{1}{2},\frac{1}{2};0,0)$
2	$(0,0;\frac{1}{2},\frac{1}{2})$	$(0,0;0,0)$ $(0,0;0,1)$ $(0,0;1,0)$	$(0,0;\frac{1}{2},\frac{1}{2})$	$(0,0;0,0)$

Table 4: The massive spin-3/2 supermultiplet $(\frac{1}{2},\frac{1}{2};\frac{1}{2},\frac{1}{2})_{\rm S}$.

2.3 The spin-3/2 multiplet

The main focus of this paper is the coupling of the massive spin-3/2 multiplet $(\frac{1}{2},\frac{1}{2};\frac{1}{2},\frac{1}{2})_{\rm S}$ which is contained twice in (2.4). Its SO(4)×SO(4) representation content is summarized in table 4. As has been discussed in the introduction, this multiplet is of particular interest for several reasons. In particular, it is the only multiplet to carry moduli of the theory. They correspond to the SO(4) × SO(4) singlet $(0,0;0,0)$ with conformal dimensions $(h_L,h_R) = (1,1)$ in table 4.

In this paper we will construct the coupling of this multiplet to the massless supergravity multiplet. In analogy to the aforementioned couplings of the spin-$\frac{1}{2}$ and spin-1 multiplet to $\mathcal{N}=8$ supergravity, a natural candidate for the effective theory might be an $\mathcal{N}=8$ gauging of the effective theory with coset space SO(8, 16)/(SO(8) × SO(16)), reproducing the correct number of 128 bosonic degrees of freedom. (The appearance of massive spin-$\frac{3}{2}$ fields would then require some analogue of the dualization taking place in the scalar/vector sector.) Let us check the representation content of table 4. It is straightforward to verify that the states of this multiplet may be lifted from a representation of the gauge group (2.2) to a representation $(\mathbf{8}_V \oplus \mathbf{8}_C, \mathbf{8}_V \oplus \mathbf{8}_C)$ of an $\mathrm{SO}(8)_L \times \mathrm{SO}(8)_R$ according to

$$
\begin{aligned}
&(\mathbf{8}_V,\mathbf{1}) \to (0,0;0,0) \oplus (0,0;0,0) \oplus (1,0;0,0) \oplus (0,1;0,0)\,,\\
&(\mathbf{8}_C,\mathbf{1}) \to \left(\frac{1}{2},\frac{1}{2};0,0\right) \oplus \left(\frac{1}{2},\frac{1}{2};0,0\right), \qquad (\mathbf{8}_S,\mathbf{1}) \to \left(\frac{1}{2},\frac{1}{2};0,0\right) \oplus \left(\frac{1}{2},\frac{1}{2};0,0\right),\\
&(\mathbf{1},\mathbf{8}_V) \to (0,0;0,0) \oplus (0,0;0,0) \oplus (0,0;1,0) \oplus (0,0;0,1)\,,
\end{aligned}
$$

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$$(\mathbf{1},\mathbf{8}_C) \to \left(0,0;\frac{1}{2},\frac{1}{2}\right) \oplus \left(0,0;\frac{1}{2},\frac{1}{2}\right), \qquad (\mathbf{1},\mathbf{8}_S) \to \left(0,0;\frac{1}{2},\frac{1}{2}\right) \oplus \left(0,0;\frac{1}{2},\frac{1}{2}\right). \tag{2.13}$$

This corresponds to an embedding of groups according to

$$\mathrm{SO}(4)_L = \mathrm{diag}[\mathrm{SO}(4) \times \mathrm{SO}(4)] \subset \mathrm{SO}(4) \times \mathrm{SO}(4) \subset \mathrm{SO}(8)_L\,, \tag{2.14}$$

and similarly for $\mathrm{SO}(4)_R$. In order to be described as a gauging of the $\mathcal{N}=8$ theory, the field content would have to be further lifted to the $(\mathbf{8}_V \oplus \mathbf{8}_C, \mathbf{16})$ of $\mathrm{SO}(8)\times\mathrm{SO}(16)$. This is only possible, if $\mathrm{SO}(8)_R$ is entirely embedded into the $\mathrm{SO}(16)$. In turn, this implies that the gravitino fields transforming in the $(\mathbf{8}_S,\mathbf{1})$ of the $\mathcal{N}=8$ theory would decompose as $(\frac{1}{2},\frac{1}{2};0,0)\oplus(\frac{1}{2},\frac{1}{2};0,0)$, in contrast to the supercharges (2.5) of the Kaluza-Klein spectrum. We conclude that the massive spin-3/2 multiplet cannot be described as a gauging of the $\mathrm{SO}(8,16)/(\mathrm{SO}(8)\times\mathrm{SO}(16))$ theory.

Rather we will find that the effective theory describing this multiplet is a maximally supersymmetric gauging of the $\mathcal{N}=16$ theory in its broken phase. Half of the supersymmetry is broken down to $\mathcal{N}=8$ and correspondingly eight gravitinos acquire mass via a super-Higgs mechanism. As a first check we observe that the total number of degrees of freedom collected in table 4 indeed equals the $16^2=256$ of the maximal theory. More specifically, according to (2.13), the total spectrum can be lifted to an $(\mathbf{8}_V\oplus\mathbf{8}_S,\mathbf{8}_V\oplus\mathbf{8}_C)$ of $\mathrm{SO}(8)_L\times\mathrm{SO}(8)_R$ and thus further to $\mathrm{SO}(16)$ according to

$$\begin{aligned} \mathbf{16} &\to (\mathbf{8}_C,\mathbf{1})\oplus(\mathbf{1},\mathbf{8}_S)\,, \\ \mathbf{128}_S &\to (\mathbf{8}_V,\mathbf{8}_V)\oplus(\mathbf{8}_S,\mathbf{8}_C)\,, \quad \mathbf{128}_C \to (\mathbf{8}_S,\mathbf{8}_V)\oplus(\mathbf{8}_V,\mathbf{8}_C)\,. \end{aligned} \tag{2.15}$$

This corresponds to the canonical embedding $\mathrm{SO}(16)\supset\mathrm{SO}(8)_L\times\mathrm{SO}(8)_R$ and an additional triality rotation. Moreover, this lifts the spectrum precisely to the $\mathbf{128}_S\oplus\mathbf{128}_C$ field content of the maximal $\mathcal{N}=16$ theory with scalar target space $G/H=E_{8(8)}/\,\mathrm{SO}(16)$. In the next section we shall describe this embedding in full detail.

3. The $\mathcal{N}=16$ supergravity action

In this section, we construct the effective three-dimensional action that describes the coupling of the massive spin-3/2 multiplet $(\frac{1}{2},\frac{1}{2};\frac{1}{2},\frac{1}{2})_{\mathrm{S}}$. We proceed in three steps. The general maximal ($\mathcal{N}=16$) supergravity in three dimensions is reviewed in section 3.1. It is given as a gauging of the maximal theory with target space $E_{8(8)}/\,\mathrm{SO}(16)$ which is entirely specified by the choice of the constant embedding tensor. In section 3.2 we determine the total gauge group G_0 of the effective theory taking into account the extra translational directions related to the presence of massive vector fields. We further describe the embedding of G_0 into the global symmetry group $E_{8(8)}$. In section 3.3 we determine the embedding tensor $\Theta_{\mathcal{MN}}$ related to this gauge group, and show that it is fixed up to four free parameters by the algebraic consistency constraints imposed by supersymmetry. Together with the general results of section 3.1 this completely determines the effective theory. We show in section 3.4 that the requirement of the existence of an AdS ground state with the correct isometry group (2.1) further fixes two of the parameters, such that the final theory has only two free parameters corresponding to the radii of the two S^3 spheres.

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3.1 Maximal supergravity in $D = 3$

We start with a brief summary of the important facts about $\mathcal{N} = 16$ supergravity in three dimensions following [13]. The spectrum consists of the supergravity multiplet which contains the vielbein e_μ^a and 16 Majorana gravitino fields ψ_μ^I, $I = 1, \ldots, 16$. The propagating matter consists of 128 real scalar fields and 128 Majorana fermions $\chi^{\dot{A}}$, $\dot{A} = 1, \ldots, 128$. They combine into an irreducible supermultiplet for the maximal amount of 32 real supercharges. The scalar fields parametrize the coset space $E_{8(8)}/\,\mathrm{SO}(16)$. Equivalently they can be represented by an $E_{8(8)}$ valued matrix $\mathcal{V}(x)$, which transforms under global $E_{8(8)}$ and local SO(16) transformations as follows:

$$\mathcal{V}(x) \to g\mathcal{V}(x)h^{-1}(x)\,, \qquad g \in E_{8(8)}\,, \quad h(x) \in \mathrm{SO}(16)\,. \tag{3.1}$$

Specifying (2.7) to the $\mathcal{N} = 16$ case, the maximal supergravity lagrangian up to quartic couplings in the fermions is given by

$$\begin{aligned}\mathcal{L} = & -\frac{1}{4}eR + \frac{1}{4}e\,\mathcal{P}^{\mu A}\mathcal{P}_\mu^A + \frac{1}{2}\varepsilon^{\mu\nu\rho}\overline{\psi}{}_\mu^I\mathcal{D}_\nu\psi_\rho^I - \frac{i}{2}e\,\bar{\chi}^{\dot{A}}\gamma^\mu\mathcal{D}_\mu\chi^{\dot{A}} - \frac{1}{2}e\bar{\chi}^{\dot{A}}\gamma^\mu\gamma^\nu\psi_\mu^I\Gamma^I_{A\dot{A}}\,\mathcal{P}_\nu^A + \\ & + \frac{1}{2}eg\,A_1^{IJ}\,\bar{\psi}_\mu{}^I\gamma^{\mu\nu}\psi_\nu^J + ieg\,A_2^{I\dot{A}}\,\bar{\chi}^{\dot{A}}\gamma^\mu\psi_\mu^I + \frac{1}{2}eg\,A_3^{\dot{A}\dot{B}}\,\bar{\chi}^{\dot{A}}\chi^{\dot{B}} + \\ & + \frac{1}{4}\,g\,\varepsilon^{\mu\nu\rho}\,A_\mu^{\mathcal{M}}\,\Theta_{\mathcal{M}\mathcal{N}}\left(\partial_\nu A_\rho^{\mathcal{N}} - \frac{1}{3}g\,\Theta_{\mathcal{K}\mathcal{L}}f^{\mathcal{N}\mathcal{L}}{}_{\mathcal{P}}\,A_\nu^{\mathcal{K}}A_\rho^{\mathcal{P}}\right) - e\,V(\mathcal{V})\,.\end{aligned} \tag{3.2}$$

As discussed in (2.8) above, the theory is entirely encoded in the symmetric constant matrix $\Theta_{\mathcal{M}\mathcal{N}}$. It describes the minimal coupling of vector fields to scalars according to

$$\mathcal{V}^{-1}D_\mu\mathcal{V} \;\equiv\; \mathcal{V}^{-1}\partial_\mu\mathcal{V} + gA_\mu^{\mathcal{M}}\,\Theta_{\mathcal{M}\mathcal{N}}\,(\mathcal{V}^{-1}t^{\mathcal{N}}\,\mathcal{V}) \;\equiv\; \frac{1}{2}\mathcal{Q}_\mu^{IJ}X^{IJ} + \mathcal{P}_\mu^A Y^A\,, \tag{3.3}$$

with X^{IJ} and Y^A labeling the 120 compact and 128 noncompact generators of $E_{8(8)}$, respectively.[3] The fermionic mass terms in (3.2) are defined as linear functions of $\Theta_{\mathcal{M}\mathcal{N}}$ via

$$\begin{aligned}A_1^{IJ} &= \frac{8}{7}\theta\delta_{IJ} + \frac{1}{7}T_{IK|JK}\,, \qquad A_2^{I\dot{A}} = -\frac{1}{7}\Gamma^J_{A\dot{A}}T_{IJ|A}\,, \\ A_3^{\dot{A}\dot{B}} &= 2\theta\delta_{\dot{A}\dot{B}} + \frac{1}{48}\Gamma^{IJKL}_{\dot{A}\dot{B}}T_{IJ|KL},\end{aligned} \tag{3.4}$$

with SO(16) gamma matrices $\Gamma^I_{A\dot{B}}$, and the T-tensor

$$T_{\mathcal{M}|\mathcal{N}} = \mathcal{V}^{\mathcal{K}}{}_{\mathcal{M}}\mathcal{V}^{\mathcal{L}}{}_{\mathcal{N}}\Theta_{\mathcal{K}\mathcal{L}}\,. \tag{3.5}$$

The scalar potential $V(\mathcal{V})$ is given by

$$V = -\frac{g^2}{8}\left(A_1^{KL}A_1^{KL} - \frac{1}{2}A_2^{K\dot{A}}A_2^{K\dot{A}}\right). \tag{3.6}$$

For later use we note the condition of stationarity of this potential:

$$\delta V = 0 \qquad \Longleftrightarrow \qquad 3A_1^{IM}A_2^{M\dot{A}} = A_3^{\dot{A}\dot{B}}A_2^{I\dot{B}}\,. \tag{3.7}$$

[3] See appendix A.1 for our $E_{8(8)}$ and SO(16) conventions.

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The quartic fermionic couplings and the supersymmetry transformations of (3.2) can be found in [13]. For consistency of the theory, the embedding tensor $\Theta_{\mathcal{M}\mathcal{N}}$ needs to satisfy two algebraic constraints. First, Θ as an element of the symmetric $E_{8(8)}$ tensor product

$$(\mathbf{248} \otimes \mathbf{248})_{\mathrm{sym}} = \mathbf{1} \oplus \mathbf{3875} \oplus \mathbf{27000}\,, \tag{3.8}$$

is required to only live in the $\mathbf{1}\oplus\mathbf{3875}$ representation, i.e. to satisfy the projection constraint $(\mathbb{P}_{\mathbf{27000}})\,\Theta \equiv 0$. Explicitly, this constraint takes the form [23]

$$\Theta_{\mathcal{M}\mathcal{N}} + \frac{1}{62}\,\eta_{\mathcal{M}\mathcal{N}}\,\eta^{\mathcal{K}\mathcal{L}}\,\Theta_{\mathcal{K}\mathcal{L}} + \frac{1}{12}\,\eta_{\mathcal{P}\mathcal{Q}}f^{\mathcal{K}\mathcal{P}}{}_{\mathcal{M}}f^{\mathcal{L}\mathcal{Q}}{}_{\mathcal{N}}\,\Theta_{\mathcal{K}\mathcal{L}} = 0\,, \tag{3.9}$$

with the $E_{8(8)}$ structure constants $f^{\mathcal{M}\mathcal{N}}{}_{\mathcal{K}}$ and Cartan-Killing form $\eta^{\mathcal{M}\mathcal{N}} = \frac{1}{60}\,f^{\mathcal{M}\mathcal{K}}{}_{\mathcal{L}}\,f^{\mathcal{N}\mathcal{L}}{}_{\mathcal{K}}$. Second, closure of the gauge group requires the $E_{8(8)}$ covariant quadratic condition

$$\Theta_{\mathcal{K}\mathcal{P}}\Theta_{\mathcal{L}(\mathcal{M}}\,f^{\mathcal{K}\mathcal{L}}{}_{\mathcal{N})} = 0\,, \tag{3.10}$$

on Θ. In turn, any solution of (3.9) and (3.10) defines a consistent maximally supersymmetric theory (3.2) in three dimensions.

It should be mentioned that although the formulation (3.2) seems to restrict vector couplings to the topological Chern-Simons term, any supergravity theory, including those with propagating Yang-Mills kinetic term can be casted into this form upon proper enhancement of the gauge group G_0 [15]. We will discuss this in more detail in the next section. In the following we will identify the particular embedding tensor $\Theta_{\mathcal{M}\mathcal{N}}$ that corresponds to the effective theory coupling the massive spin-3/2 multiplet on the $AdS_3\times S^3\times S^3$ background.

3.2 The gauge group G_0

In this section, we will identify the full gauge group of the effective three-dimensional theory and determine its embedding into the global $E_{8(8)}$ symmetry group of the ungauged theory. To this end we need to first review the general structure of gauge groups in three dimensions. It has been shown in [15] that generically the gauge group of a three-dimensional Chern-Simons gauged supergravity (3.2) is of the form

$$G_0 = G_{\mathrm{c}} \ltimes (\hat{T}_p, T_\nu)\,. \tag{3.11}$$

In our case, G_{c} denotes the compact gauge group (2.2) which from the Kaluza-Klein origin of the theory is expected to be realized by propagating vector fields. In the Chern-Simons formulation given above, this compact factor needs to be amended by the nilpotent translation group T_ν whose $\nu = \dim G_{\mathrm{c}}$ generators transform in the adjoint representation of G_{c}. This allows an alternative formulation of the theory (3.2) in which part of the scalar sector is redualized into propagating vector fields gauging the group G_{c} which accordingly appear with a conventional Yang-Mills term. In other words, a Yang-Mills gauged theory with gauge group G_{c} is on-shell equivalent to a Chern-Simons gauged supergravity theory with gauge group $G_{\mathrm{c}} \ltimes \mathcal{T}_\nu$. The third factor $\hat{T}_p$ in (3.11) is spanned by p nilpotent translations transforming in some representation of G_{c} and closing into T_ν. This part of the gauge group is completely broken in the vacuum and gives rise to p massive vector fields.

Specifically, the algebra underlying (3.11) reads

$$
\begin{aligned}
&[\mathcal{J}^m, \mathcal{J}^n] = f^{mn}{}_k\, \mathcal{J}^k\,, &&[\mathcal{J}^m, \mathcal{T}^{\underline{n}}] = f^{mn}{}_k\, \mathcal{T}^{\underline{k}}\,, &&[\mathcal{T}^{\underline{m}}, \mathcal{T}^{\underline{n}}] = 0\,,\\
&[\mathcal{J}^m, \hat{\mathcal{T}}^\alpha] = t^{m\alpha}{}_\beta\, \hat{\mathcal{T}}^\beta\,, &&[\hat{\mathcal{T}}^\alpha, \hat{\mathcal{T}}^\beta] = t^{\alpha\beta}{}_m\, \mathcal{T}^{\underline{m}}\,, &&[\mathcal{T}^{\underline{m}}, \hat{\mathcal{T}}^\alpha] = 0\,,
\end{aligned}
\tag{3.12}
$$

with $\mathcal{J}^m$, $\mathcal{T}^{\underline{n}}$, and $\hat{\mathcal{T}}^\alpha$ generating $G_{\rm c}$, T_ν, and $\hat{T}_p$, respectively. The $f^{mn}{}_k$ are the structure constants of $G_{\rm c}$ while the $t^{m\alpha}{}_\beta$ denote the representation matrices for the $\hat{\mathcal{T}}^\alpha$. Indices $m, n, \ldots$ are raised/lowered with the Cartan-Killing form of $G_{\rm c}$; raising/lowering of indices α, β requires a symmetric $G_{\rm c}$ invariant tensor $\kappa^{\alpha\beta}$.

To begin with, we have to reconcile the structure (3.11) with the spectrum collected in table 4. With $G_{\rm c} = {\rm SO}(4)_L \times {\rm SO}(4)_R$ from (2.2), T_ν transforms in the adjoint representation $(1,0;0,0) \oplus (0,1;0,0) \oplus (0,0;1,0) \oplus (0,0;0,1)$. table 4 exhibits 34 additional massive vector fields, transforming in the $2\cdot(\frac{1}{2},\frac{1}{2};\frac{1}{2},\frac{1}{2}) \oplus 2\cdot(0,0;0,0)$ of $G_{\rm c}$. In total, we thus expect a gauge group $G_0 = G_{\rm c} \ltimes (\hat{T}_{34}, T_{12})$ of dimension $\dim G_0 = 12 + 12 + 34 = 58$. Next, we have to identify this group within $E_{8(8)}$. To this end, it proves useful to first consider the embedding of $G_{\rm c}$ into $E_{8(8)}$ according to the decompositions

$$
E_{8(8)} \;\supset\; \left\{ \begin{matrix} \supset\; {\rm SO}(16) \;\supset \\[2ex] \supset {\rm SO}(8,8) \supset \end{matrix} \right\} \;\supset\; {\rm SO}(8)_L \times {\rm SO}(8)_R \;\supset\; {\rm SO}(4)_L \times {\rm SO}(4)_R\,,
\tag{3.13}
$$

with the two embeddings of ${\rm SO}(8)_L \times {\rm SO}(8)_R$ given by

$$
\begin{aligned}
{\rm SO}(16):&\quad \mathbf{16} \to (\mathbf{8}_C, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{8}_S)\,, \quad \mathbf{128}_S \to (\mathbf{8}_V, \mathbf{8}_V) \oplus (\mathbf{8}_S, \mathbf{8}_C)\,,\\
{\rm SO}(8,8):&\quad \mathbf{16} \to (\mathbf{8}_V, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{8}_V)\,, \quad \mathbf{128}_S \to (\mathbf{8}_C, \mathbf{8}_S) \oplus (\mathbf{8}_S, \mathbf{8}_C)\,.
\end{aligned}
\tag{3.14}
$$

Accordingly, the group $E_{8(8)}$ decomposes as

$$
\mathbf{248} \;\to\; \{(\mathbf{28}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{28}) \oplus (\mathbf{8}_C, \mathbf{8}_S)\} \;\oplus\; \{(\mathbf{8}_V, \mathbf{8}_V) \oplus (\mathbf{8}_S, \mathbf{8}_C)\}\,,
\tag{3.15}
$$

and further according to (2.13). Here curly brackets indicate the splitting into its compact and noncompact part and $\mathbf{28} = \mathbf{8} \wedge \mathbf{8}$. We have already discussed in section 2.3 that with this embedding the noncompact part of $E_{8(8)}$ precisely reproduces the bosonic spectrum of table 4.

In order to identify the embedding of the full gauge group $G_0 = G_{\rm c} \ltimes (\hat{T}_{34}, T_{12})$ we further consider the decomposition of $E_{8(8)}$ according to

$$
E_{8(8)} \supset {\rm SO}(8,8) \supset {\rm SO}(6,6) \times {\rm SO}(2,2) \supset {\rm SO}(6,6) \times {\rm SO}(1,1) \times {\rm SO}(1,1)\,,
\tag{3.16}
$$

and its grading w.r.t. these two ${\rm SO}(1,1)$ factors which is explicitly given in table 5. From this table we can infer that properly identifying

$$
G_{\rm c} \;\subset\; \mathbf{66}^0_0\,, \qquad T_{12} \;=\; \mathbf{12}^{+1}_0\,, \qquad \hat{T}_{34} \;\subset\; \mathbf{32}^{+1/2}_{-1/2} \;\oplus\; \overline{\mathbf{32}}^{+1/2}_{+1/2} \;\oplus\; \mathbf{1}^{+1}_{-1} \;\oplus\; \mathbf{1}^{+1}_{+1}\,,
\tag{3.17}
$$

precisely reproduces the desired algebra structure (3.12). We have thus succeeded in identifying the algebra $\mathfrak{g}_0$ underlying the full gauge group $G_0 = G_{\rm c} \ltimes (\hat{T}_{34}, T_{12})$, which is entirely embedded in the "upper light cone" of table 5. In the next section, we will explicitly construct the embedding tensor $\Theta_{\mathcal{MN}}$ projecting onto this algebra, and show that it is indeed compatible with the algebraic constraints (3.9), (3.10) imposed by supersymmetry.

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$$\begin{array}{ccccccc}
\mathbf{1}^{+1}_{-1} & & & \mathbf{12}^{+1}_{0} & & & \mathbf{1}^{+1}_{+1} \\
& \mathbf{32}^{+1/2}_{-1/2} & & & & \overline{\mathbf{32}}^{+1/2}_{+1/2} & \\
\mathbf{12}^{0}_{-1} & & & \mathbf{66}^{0}_{0}+\mathbf{1}^{0}_{0}+\mathbf{1}^{0}_{0} & & & \mathbf{12}^{0}_{+1} \\
& \overline{\mathbf{32}}^{-1/2}_{-1/2} & & & & \mathbf{32}^{-1/2}_{+1/2} & \\
\mathbf{1}^{-1}_{-1} & & & \mathbf{12}^{-1}_{0} & & & \mathbf{1}^{-1}_{+1}
\end{array}$$

Table 5: Grading of $E_{8(8)}$ according to $\mathrm{SO}(6,6)\times\mathrm{SO}(1,1)\times\mathrm{SO}(1,1)$. For later reference, we denote by $\mathrm{SO}(1,1)_a$ the factor responsible for the grading from left to right and by $\mathrm{SO}(1,1)_b$ the factor responsible for the grading from top to bottom.

3.3 The embedding tensor

In this section, we will explicitly construct the embedding tensor $\Theta_{\mathcal{MN}}$ projecting onto the Lie algebra $\mathfrak{g}_0$ of the desired gauge group $G_0 = G_{\rm c} \ltimes (\hat{T}_{34}, T_{12})$ identified in the previous section. The embedding tensor then uniquely defines the effective action (3.2). We start from the $\mathrm{SO}(4)\times\mathrm{SO}(4)$ basis of $E_{8(8)}$ defined in appendix A.4. In this basis, the grading of table 5 refers to the charges of the generators $X^{0\hat{0}}$ and $X^{\bar{0}\hat{\bar{0}}}$. We further denote the generators of G_c and T_{12} within table 5 as

$$\mathbf{66}^{0}_{0} \supset \mathbf{3}^{+}_{L}\oplus\mathbf{3}^{-}_{L}\oplus\mathbf{3}^{+}_{R}\oplus\mathbf{3}^{-}_{R}\,, \qquad \text{and} \quad \mathbf{12}^{+1}_{0} \;=\; \hat{\mathbf{3}}^{+}_{L}\oplus\hat{\mathbf{3}}^{-}_{L}\oplus\hat{\mathbf{3}}^{+}_{R}\oplus\hat{\mathbf{3}}^{-}_{R}\,, \tag{3.18}$$

respectively, with the labels L, R, $\pm$ referring to the four factors of (2.2), i.e. $\mathbf{3}^{+}_{L} = (1,0;0,0)$, $\mathbf{3}^{-}_{L} = (0,1;0,0)$, etc. Similarly, we will identify the generators of $\hat{T}_{34}$ among

$$\mathbf{32}^{+1/2}_{-1/2} \equiv \mathbf{16}^{(1)}_{-}\oplus\mathbf{16}^{(2)}_{-}\,, \qquad \overline{\mathbf{32}}^{+1/2}_{+1/2} \equiv \mathbf{16}^{(1)}_{+}\oplus\mathbf{16}^{(2)}_{+}\,, \qquad \mathbf{1}^{+1}_{-1}\equiv\mathbf{1}_{-}\,, \quad \mathbf{1}^{+1}_{+1}\equiv\mathbf{1}_{+}\,, \tag{3.19}$$

where $\mathbf{16}$ denotes the $(\frac{1}{2},\frac{1}{2};\frac{1}{2},\frac{1}{2})$ of $\mathrm{SO}(4)\times\mathrm{SO}(4)$, and we use subscripts $(1),(2)$, $\pm$ in order to distinguish the identical representations. The split of the $\mathbf{32}$ representations into two copies of the $\mathbf{16}$ is chosen such that the algebra closes according to

$$\begin{aligned}
\left[\mathbf{16}^{(1)}_{+},\mathbf{16}^{(1)}_{-}\right] &\subset \hat{\mathbf{3}}^{-}_{L}\oplus\hat{\mathbf{3}}^{-}_{R}\,, \qquad & \left[\mathbf{16}^{(2)}_{+},\mathbf{16}^{(2)}_{-}\right] &\subset \hat{\mathbf{3}}^{-}_{L}\oplus\hat{\mathbf{3}}^{-}_{R}\,, \\
\left[\mathbf{16}^{(1)}_{+},\mathbf{16}^{(2)}_{-}\right] &\subset \hat{\mathbf{3}}^{+}_{L}\oplus\hat{\mathbf{3}}^{+}_{R}\,, \qquad & \left[\mathbf{16}^{(2)}_{+},\mathbf{16}^{(1)}_{-}\right] &\subset \hat{\mathbf{3}}^{+}_{L}\oplus\hat{\mathbf{3}}^{+}_{R}\,.
\end{aligned} \tag{3.20}$$

The embedding tensor $\Theta_{\mathcal{MN}}$ is an object in the symmetric tensor product of two adjoint representations of $E_{8(8)}$. It projects onto the Lie algebra of the gauge group according to $\mathfrak{g}_0 = \langle X_{\mathcal{M}} \equiv \Theta_{\mathcal{MN}}\, t^{\mathcal{N}}\rangle$. We start from the most general ansatz for $\Theta_{\mathcal{MN}}$ that has entries only on the generators (3.18), (3.19). Since the $\Theta_{\mathcal{MN}}$ relevant for our theory moreover is an $\mathrm{SO}(4)\times\mathrm{SO}(4)$ invariant tensor, it can only have non-vanishing entries contracting coinciding representations, e.g. $\Theta_{\mathbf{3}^{+}_{L},\hat{\mathbf{3}}^{+}_{L}}$, $\Theta_{\mathbf{16}^{(1)}_{+},\mathbf{16}^{(2)}_{-}}$, etc. Using computer algebra (Mathematica), we can then implement the algebraic constraint (3.9). As one of the main results of this paper, we find that this constraint determines the embedding tensor Θ with these properties up to five free constants γ, β_1, β_2, β_3, β_4, in terms of which it takes the form[4]

$$
\begin{aligned}
\Theta_{\mathbf{3}_L^+,\hat{\mathbf{3}}_L^+} &= \beta_1\,, \qquad \Theta_{\mathbf{3}_L^-,\hat{\mathbf{3}}_L^-} = \beta_2\,, \qquad \Theta_{\mathbf{3}_R^+,\hat{\mathbf{3}}_R^+} = \beta_3\,, \qquad \Theta_{\mathbf{3}_R^-,\hat{\mathbf{3}}_R^-} = \beta_4\,,\\
\Theta_{\mathbf{1}_+,\mathbf{1}_-} &= \Theta_{\hat{\mathbf{3}}_R^+,\hat{\mathbf{3}}_R^+} = \Theta_{\hat{\mathbf{3}}_R^-,\hat{\mathbf{3}}_R^-} = -\Theta_{\hat{\mathbf{3}}_L^+,\hat{\mathbf{3}}_L^+} = -\Theta_{\hat{\mathbf{3}}_L^-,\hat{\mathbf{3}}_L^-} = \gamma\,,\\
\Theta_{\mathbf{16}_+^{(1)},\mathbf{16}_-^{(1)}} &= -\frac{1}{32\sqrt{2}}(\beta_2+\beta_4)\,, \qquad \Theta_{\mathbf{16}_+^{(1)},\mathbf{16}_-^{(2)}} = -\frac{1}{32\sqrt{2}}(\beta_1+\beta_3)\,,\\
\Theta_{\mathbf{16}_+^{(2)},\mathbf{16}_-^{(1)}} &= \frac{1}{32\sqrt{2}}(\beta_1-\beta_3)\,, \qquad \Theta_{\mathbf{16}_+^{(2)},\mathbf{16}_-^{(2)}} = -\frac{1}{32\sqrt{2}}(\beta_2-\beta_4)\,.
\end{aligned}
\tag{3.21}
$$

A priori, it seems quite surprising that the constraint (3.9) still leaves five free constants in Θ — the **27000** representation of $E_{8(8)}$ gives rise to 1552 different $\mathrm{SO}(4)\times\mathrm{SO}(4)$ representations that are separately imposed as constraints on our general ansatz for Θ.

In order to satisfy the full set of consistency constraints it remains to impose the quadratic constraint (3.10) on the embedding tensor $\Theta_{\mathcal{MN}}$. Again using computer algebra, we can compute the form of this constraint for the embedding tensor (3.21) and find that it reduces to a single condition on the parameters:

$$
\beta_1^2+\beta_2^2 = \beta_3^2+\beta_4^2\,. \tag{3.22}
$$

This suggests the parametrization as

$$
\beta_1 = \kappa\sin\alpha_1\,, \qquad \beta_2 = \kappa\cos\alpha_1\,, \qquad \beta_3 = \kappa\sin\alpha_2\,, \qquad \beta_4 = \kappa\cos\alpha_2\,. \tag{3.23}
$$

Altogether we have shown, that there is a four parameter family of maximally supersymmetric theories, described by the embedding tensor (3.21), which satisfies all the consistency constraints (3.9), (3.10).

For generic values of the parameters, one verifies that the rank of the induced gauge group is indeed 58 as expected.[5] In particular, (3.21), (3.22) imply that on the block of **16** representations one finds

$$
\begin{aligned}
\Theta_{\mathcal{MN}}\, t^{\mathcal{M}}\otimes t^{\mathcal{N}}\Big|_{\mathbf{16}} &= -\frac{\kappa}{16\sqrt{2}}\left(\mathbf{16}_+^{(1)}\cos\left(\frac{1}{2}\,(\alpha_1-\alpha_2)\right) - \mathbf{16}_+^{(2)}\sin\left(\frac{1}{2}\,(\alpha_1-\alpha_2)\right)\right)\otimes\\
&\quad\otimes\left(\mathbf{16}_-^{(1)}\cos\left(\frac{1}{2}\,(\alpha_1+\alpha_2)\right) - \mathbf{16}_-^{(2)}\sin\left(\frac{1}{2}\,(\alpha_1+\alpha_2)\right)\right).
\end{aligned}
\tag{3.24}
$$

This implies that out of the 64 generators $\mathbf{16}_\pm^{(1)}$, $\mathbf{16}_\pm^{(2)}$, only the 32 combinations

$$
\begin{aligned}
\mathbf{16}_+ &\equiv \mathbf{16}_+^{(1)}\cos\left(\frac{1}{2}\,(\alpha_1-\alpha_2)\right) - \mathbf{16}_+^{(2)}\sin\left(\frac{1}{2}\,(\alpha_1-\alpha_2)\right),\\
\mathbf{16}_- &\equiv \mathbf{16}_-^{(1)}\cos\left(\frac{1}{2}\,(\alpha_1+\alpha_2)\right) - \mathbf{16}_-^{(2)}\sin\left(\frac{1}{2}\,(\alpha_1+\alpha_2)\right),
\end{aligned}
\tag{3.25}
$$

[4]Here we have used a somewhat symbolic notation for Θ, indicating just the multiples of the identity matrix that Θ takes in the various blocks.

[5]Let us note that the degenerate case $\kappa = 0$ induces a theory with 14-dimensional nilpotent abelian gauge group, as can be seen from (3.21). This particular gauge group had already been identified in [18].

form part of the gauge group. These correspond to the $2{\cdot}(\frac{1}{2},\frac{1}{2};\frac{1}{2},\frac{1}{2})$ generators in $\hat{T}_{34}$. The complete gauge algebra spanned by the generators $X_{\mathcal{M}} \equiv \Theta_{\mathcal{MN}}\, t^{\mathcal{N}}$ is precisely of the form anticipated in (3.12).

Let us stress another important property of the embedding tensor (3.21): it is a singlet not only under the $\mathrm{SO}(4)\times\mathrm{SO}(4)$, but also under the $\mathrm{SO}(1,1)_a$ generating the grading from left to right in table 5. In other words, the resulting Θ contracts only representations for which these particular charges add up to zero. As a consequence the gauged supergravity (3.2) in addition to the local gauge symmetry $G_0 = G_{\mathrm{c}} \ltimes (\hat{T}_{34}, T_{12})$ is invariant under the action of the global symmetry $\mathrm{SO}(1,1)_a$. We will discuss the physical consequences of this extra symmetry in section 3.5 below.

3.4 Ground state and isometries

In the previous section we have found a four-parameter family of solutions $\Theta_{\mathcal{MN}}$ (3.21) to the algebraic constraints (3.9), (3.10) compatible with the gauge algebra $G_0 = G_{\mathrm{c}} \ltimes (\hat{T}_{34}, T_{12})$. We will now show that the four free parameters γ, κ, α_1, α_2, can be adjusted such that the theory admits an $\mathcal{N}=(4,4)$ supersymmetric AdS ground state, leaving only two free parameters that correspond to the the radii of the two S^3 spheres. Furthermore, expanding the action (3.2) around this ground state precisely reproduces the spectrum of table 4.

In order to show, that the lagrangian (3.2) admits an AdS ground state, we first have to check the condition (3.7) equivalent to the existence of a stationary point of the scalar potential (3.6). For this in turn we have to compute the tensors A_1, A_2 and A_3 (3.4) from the T-tensor (3.5) evaluated at the ground state $\mathcal{V} = I$. At this point, the T-tensor coincides with the embedding tensor (3.21). The only technical problem is the translation from Θ (3.21) in the $\mathrm{SO}(8,8)$ basis of appendix A.2 into the $\mathrm{SO}(16)$ basis of appendix A.1 in which the tensors A_1, A_2 and A_3 are defined.

It follows from (3.21) that Θ is traceless, $\theta = 0$, and moreover that all components of Θ, which mix bosonic and spinorial parts, like $\Theta_{ab|\alpha\dot\beta}$, vanish. As a consequence, the tensor A_1 is block-diagonal, and its explicit form turns out to be

$$A_1^{IJ} = \frac{1}{7}\begin{pmatrix} 2\Theta_{\dot\alpha\gamma|\dot\beta\gamma} + \overline{\Gamma}^{\hat a\hat b}_{\dot\alpha\dot\gamma}\overline{\Gamma}^{\hat c\hat d}_{\dot\beta\dot\gamma}\Theta_{\hat a\hat b|\hat c\hat d} & 0 \\ 0 & 2\Theta_{\dot\gamma\alpha|\dot\gamma\beta} + \Gamma^{ab}_{\alpha\gamma}\Gamma^{cd}_{\beta\gamma}\Theta_{ab|cd} \end{pmatrix}, \tag{3.26}$$

with $\mathrm{SO}(8)$ Γ-matrices $\Gamma^a_{\alpha\dot\beta}$, see appendix A.3 for details. Similarly, A_2 and A_3 are also block-diagonal and can be written as

$$A_2^{I\dot A} = -\frac{1}{7}\begin{pmatrix} 2\Gamma^a_{\gamma\dot\epsilon}\Theta_{\dot\alpha\gamma|\beta\dot\epsilon} - \Gamma^{\hat b}_{\beta\dot\gamma}\overline{\Gamma}^{\hat c\hat d}_{\dot\alpha\dot\gamma}\Theta_{\hat c\hat d|a\hat b} & 0 \\ 0 & 2\Gamma^{\hat a}_{\delta\dot\gamma}\Theta_{\dot\gamma\alpha|\delta\dot\beta} + \Gamma^{cd}_{\alpha\gamma}\Gamma^b_{\gamma\dot\beta}\Theta_{cd|b\hat a} \end{pmatrix}, \tag{3.27}$$

and

$$A_3^{\dot A\dot B} = \begin{pmatrix} \delta^{\alpha\beta}\Theta_{a\hat c|b\hat c} + 2\delta^{ab}\Theta_{\alpha\dot\gamma|\beta\dot\gamma} & 0 \\ 0 & \delta^{\dot\alpha\dot\beta}\Theta_{c\hat a|c\hat b} + 2\delta^{\hat a\hat b}\Theta_{\gamma\dot\alpha|\gamma\dot\beta} \end{pmatrix}. \tag{3.28}$$

Using these tensors one can now check that the ground state condition (3.7) is fulfilled if the parameters of (3.21) satisfy

$$\kappa^2 = 16\gamma^2 \,. \tag{3.29}$$

Moreover, using (3.6) the value of the scalar potential at the ground state, i.e. the cosmological constant, can be computed and consistently comes out to be negative $V = -g^2/2$, i.e. the AdS length is given by $L_0 = 1/g$.

In the following, we will absorb κ into the global coupling constant g and set $\gamma = 1/4$ in accordance with (3.29). As a result, there remains a two-parameter family of supergravities admitting an AdS ground state. Let us now determine the number of unbroken supercharges in the ground state. It is derived from the Killing spinor equations

$$\delta\psi^I_\mu = D_\mu\epsilon^I + igA_1^{IJ}\gamma_\mu\epsilon^J \equiv 0\,, \qquad \delta\chi^{\dot{A}} = gA_2^{I\dot{A}}\epsilon^I \equiv 0\,. \tag{3.30}$$

As has been shown in [13], the number of solutions to (3.30) and thus the number of preserved supersymmetries is given by the number of eigenvalues α_i of the tensor A_1^{IJ} with $|\alpha_i|gL_0 = 1/2$. Computing these eigenvalues from the explicit form of (3.26) we find that the tensor A_1^{IJ} may be diagonalized as

$$A_1^{IJ} = \mathrm{diag}\left\{-\frac{3}{2},-\frac{3}{2},-\frac{3}{2},-\frac{3}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{3}{2},\frac{3}{2},\frac{3}{2}\right\}. \tag{3.31}$$

From this, we infer that the AdS ground state of the theory indeed preserves $\mathcal{N} = (4,4)$ supersymmetries, as expected. The other eight gravitinos become massive through a super-Higgs mechanism [17, 24]. This implies that due to the broken supersymmetries eight of the spin-1/2 fermions

$$\eta^I \equiv A_2^{I\dot{A}}\chi^{\dot{A}}\,, \tag{3.32}$$

transform by a shift under supersymmetry and act as Goldstone fermions that get eaten by the gravitino fields which in turn become massive propagating spin-3/2 fields. With the relation

$$|\Delta - 1| = |m|\,L_0\,, \tag{3.33}$$

between the AdS masses m and conformal dimensions Δ of fermions and self-dual massive vectors in three dimensions, (3.31) implies that the massive gravitinos correspond to operators with conformal weights $(\frac{1}{2},2)$ and $(2,\frac{1}{2})$, in precise agreement with the spectrum of table 4.

To compute the physical masses for the spin-1/2 fermions, we observe from (3.2) that their mass matrix is given by $gA_3^{\dot{A}\dot{B}}$, except for the eight eigenvalues that correspond to the Goldstone fermions (3.32). From the explicit form (3.28) one computes the spin-1/2 masses and verifies using (3.33) that they coincide with those of table 4. Finally, we may check the mass spectrum for the spin-1 fields. Their mass matrix is given by Θ_{AB}, the projection of the embedding tensor onto the non-compact part of the algebra [17]. From (3.21) one finds by explicit computation for these eigenvalues 46 non-vanishing values in precise accordance with table 4.

Altogether, we have shown the existence of a new family of gauged maximally supersymmetric theories in $D = 3$, which are parametrized by the two free parameters α_1 and α_2 and the overall gauge coupling constant g. These theories admit an $\mathcal{N} = (4,4)$ supersymmetric AdS_3 ground state and linearizing the field equations around this ground state reproduces the correct spectrum of table 4. In particular, this spectrum does not depend on the particular values of α_1 and α_2. One may still wonder about the meaning of these two parameters. From the point of view of the Kaluza-Klein reduction the only relevant parameter is the ratio α of the two spheres radii, which enters the superalgebra (2.1). Let us thus compute the background isometry group by expanding the supersymmetry algebra

$$\{ \delta_{\epsilon_1}, \delta_{\epsilon_2} \} = (\bar{\epsilon}_1^I \epsilon_2^J)\, \mathcal{V}^{\mathcal{M}}{}_{IJ}\, \Theta_{\mathcal{M}\mathcal{N}}\, t^{\mathcal{N}} + \cdots \,, \tag{3.34}$$

around the ground state $\mathcal{V} = I$. The conserved supercharges ϵ^I are the eigenvectors of A_1 from (3.31) to the eigenvalues $\pm 1/2$ where the different signs correspond to the split into left and right supercharges according to (2.1). Correspondingly, the algebra (3.34) splits into two parts, L and R, with anticommutators

$$\{G^i_{-1/2\,L,R}, G^j_{1/2\,L,R}\} = 4\left(\frac{1}{1+\alpha_{L,R}}\,\tau^{+ij}_{kl}\, J^{+kl}_{L,R} + \frac{\alpha_{L,R}}{1+\alpha_{L,R}}\,\tau^{-ij}_{kl}\, J^{-kl}_{L,R}\right) + \cdots \,, \tag{3.35}$$

where $\tau^{\pm ij}_{kl} \equiv \delta^{ij}_{kl} \pm \frac{1}{2}\epsilon_{ijkl}$ denote the projectors onto selfdual and anti-selfdual generators of $\mathrm{SO}(4)_{L,R}$ corresponding to the split $\mathrm{SO}(4) = \mathrm{SO}(3)^+ \times \mathrm{SO}(3)^-$. This coincides with the anticommutators of the superalgebra $D^1(2,1;\alpha_{L,R})$ [19]. Specifically, we find the relation

$$\alpha_L = \tan\alpha_1 \,, \qquad \alpha_R = \tan\alpha_2 \,, \tag{3.36}$$

to the parameters (3.23) of the embedding tensor. This shows that the three-parameter family of theories constructed in this section exhibits the background isometry group

$$D^1(2,1;\alpha_L)_L \times D^1(2,1;\alpha_R)_R \,. \tag{3.37}$$

The theories related to the Kaluza-Klein compactification on $AdS_3 \times S^3 \times S^3$ are thus given by further restricting $\alpha_L = \alpha_R \equiv \alpha$ where this parameter corresponds to the ratio of radii of the two spheres.

Putting everything together, we have shown that the effective supergravity action describing the field content of table 4 is given by the lagrangian (3.2) with the following particular form of the embedding tensor $\Theta_{\mathcal{M}\mathcal{N}}$

$$\begin{aligned}
\Theta_{\mathbf{3}^+_L,\hat{\mathbf{3}}^+_L} &= \Theta_{\mathbf{3}^+_R,\hat{\mathbf{3}}^+_R} = \frac{\alpha}{\sqrt{1+\alpha^2}}\,, \qquad \Theta_{\mathbf{3}^-_L,\hat{\mathbf{3}}^-_L} = \Theta_{\mathbf{3}^-_R,\hat{\mathbf{3}}^-_R} = \frac{1}{\sqrt{1+\alpha^2}}\,, \\
\Theta_{\mathbf{1}_+,\mathbf{1}_-} &= \Theta_{\hat{\mathbf{3}}^+_R,\hat{\mathbf{3}}^+_R} = \Theta_{\hat{\mathbf{3}}^-_R,\hat{\mathbf{3}}^-_R} = -\,\Theta_{\hat{\mathbf{3}}^+_L,\hat{\mathbf{3}}^+_L} = -\,\Theta_{\hat{\mathbf{3}}^-_L,\hat{\mathbf{3}}^-_L} = \frac{1}{4}\,, \\
\Theta_{\mathbf{16}^{(1)}_+,\mathbf{16}^{(1)}_-} &= -\frac{1}{16\sqrt{2}}\,\frac{1}{\sqrt{1+\alpha^2}}\,, \qquad \Theta_{\mathbf{16}^{(1)}_+,\mathbf{16}^{(2)}_-} = -\frac{1}{16\sqrt{2}}\,\frac{\alpha}{\sqrt{1+\alpha^2}}\,.
\end{aligned} \tag{3.38}$$

We have verified that this tensor indeed represents a solution of the algebraic consistency constraints (3.9), (3.10). The resulting theory admits an $\mathcal{N} = (4,4)$ supersymmetric AdS_3 ground state with background isometry group (2.1) at which half of the 16 supersymmetries are spontaneously broken and the spectrum of table 4 is reproduced via a supersymmetric version of the Brout-Englert-Higgs effect.

3.5 The scalar potential for the gauge group singlets

We have identified the gauged supergravity theory, whose broken phase describes the coupling of the massive spin-3/2 multiplet $(\frac{1}{2}, \frac{1}{2}; \frac{1}{2}, \frac{1}{2})_{\rm S}$ to the supergravity multiplet. In particular, the scalar potential (3.6) of the effective three-dimensional theory is completely determined in terms of the embedding tensor (3.38). In the holographic context this scalar potential carries essential information about the boundary conformal field theory, in particular about higher point correlation functions and about deformations and renormalization group flows. Explicit computation of the full potential is a highly nontrivial task, as it is a function on the 128-dimensional target space $E_{8(8)}/\,{\rm SO}(16)$. For concrete applications it is often sufficient to evaluate this potential on particular subsectors of the scalar manifold.

As an example, let us in this final section evaluate the potential on the gauge group singlets. From table 4 we read off that there are two scalar fields that are singlets under the ${\rm SO}(4)_L \times {\rm SO}(4)_R$ gauge group. Let us denote them by ϕ_1 and ϕ_2. They are dual to a marginal and an irrelevant operator of conformal dimension $(1, 1)$ and $(2, 2)$, respectively. In particular, the scalar ϕ_1 corresponds to a modulus of the theory. In order to determine the explicit dependence of the scalar potential on these fields, we parametrize the scalar $E_{8(8)}$ matrix $\mathcal{V}$ as

$$\mathcal{V} = \exp\left(\phi_1 X^{0\hat{0}} + \phi_2 X^{\bar{0}\hat{\bar{0}}}\right), \tag{3.39}$$

where $X^{0\hat{0}}$ and $X^{\bar{0}\hat{\bar{0}}}$ are the generators of the ${\rm SO}(1, 1)_a$ and ${\rm SO}(1, 1)_b$ of table 5, respectively. The potential is obtained by computing with this parametrization the T-tensor from (3.5), (3.38), splitting it into the tensors A_1 and A_2 according to (3.4) and inserting the result into (3.6).

The computation is simplified by first transforming the two singlets into a basis where their adjoint action is diagonal, such that their exponentials can be easily computed and afterwards transforming back to the SO(16) basis of appendix A.1. It becomes now crucial that the embedding tensor Θ is invariant under ${\rm SO}(1, 1)_a$ and thus under the adjoint action of $X^{0\hat{0}}$ as we found in section 3.3 above. This implies, that the T-tensor (3.5) is in fact completely independent of ϕ_1. In turn, neither the fermionic mass terms nor the scalar potential carries an explicit dependence on ϕ_1. This scalar thus enters the theory only through its kinetic term, the dual operator is truly marginal.

The scalar potential (3.6) evaluated on the gauge group singlets is finally given as a function of ϕ_2 as

$$V(\phi_1, \phi_2) = \frac{g^2}{4} e^{2\phi_2} \left(-2 + e^{2\phi_2}\right). \tag{3.40}$$

The profile is plotted in figure 1. Its explicit form shows that the theory has no other ground state which preserves the full ${\rm SO}(4) \times {\rm SO}(4)$ symmetry.

4. Conclusions

In this paper we have described the effective supergravity actions for the lowest massive Kaluza-Klein multiplets in the Kaluza-Klein towers on $AdS_3 \times S^3 \times S^3$. In particular, we have constructed the coupling of the massive spin-3/2 multiplet $(\frac{1}{2}, \frac{1}{2}; \frac{1}{2}, \frac{1}{2})_{\rm S}$ to the

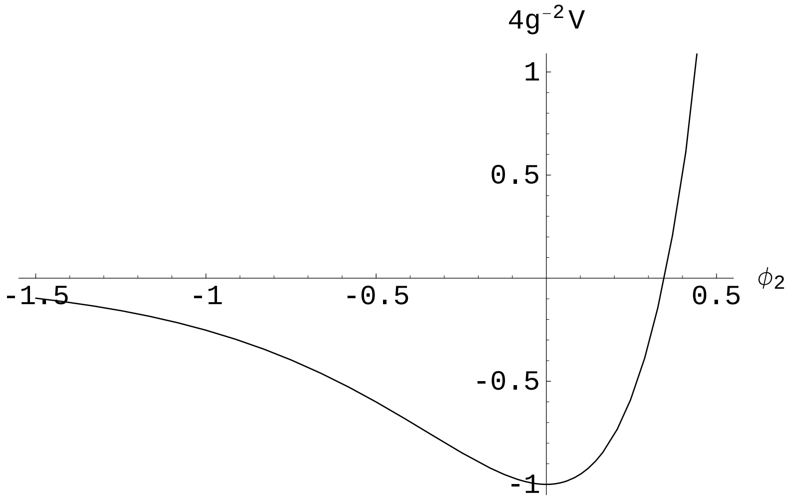

Figure 1: The scalar potential for the gauge group singlets.

massless supergravity multiplet. The corresponding three-dimensional effective theory is the maximal gauged supergravity (3.2) which is completely specified by the choice of the embedding tensor (3.38). We have verified that this tensor indeed provides a solution to the algebraic consistency constraints (3.9), (3.10). Specifically we have shown that these constraints together with the requirement of the existence of an AdS_3 ground state with the correct isometries uniquely determine the theory up to two constant parameters corresponding to the radii of the two S^3 spheres. The effective theory comes with a local gauge symmetry $\mathrm{SO}(4)_L \times \mathrm{SO}(4)_R \ltimes (\hat{T}_{34}, T_{12})$ whose 46-dimensional translational part is broken at the AdS ground state and gives rise to the massive spin-1 fields of table 4. Similarly, half of the supersymmetries are spontaneously broken at the ground state, and the full spectrum precisely reproduces the field content of table 4.

The effective theory constructed in this paper offers various routes to shed further light onto the holographic boundary CFT. It will be of particular interest to evaluate the scalar potential (3.6) on different subsectors of the supergravity scalars. This provides further insight into the structure of deformations of the boundary theory. For instance, the potential structure on the scalars in the $(0,0;1,0) \oplus (0,0;0,1)$ representation will encode further information on the marginal deformations discussed in [25] that break the superconformal $\mathcal{N} = (4,4)$ symmetry down to $\mathcal{N} = (4,0)$. Another interesting class of deformations of the boundary theory are the relevant deformations by the dimension 1 operators in the $(\frac{1}{2}, \frac{1}{2}; \frac{1}{2}, \frac{1}{2})$ representation.

A natural question about the construction presented here is the extension of the analysis to the rest of the Kaluza-Klein spectrum. Since the spin-3/2 multiplet is realized as a maximally supersymmetric theory, it is impossible to couple further fields to it: the scalar target space of the maximal theory is the unique coset space $E_{8(8)}/\,\mathrm{SO}(16)$ whose 128 degrees of freedom match (and are exhausted by) the bosonic content of the spin-3/2 multiplet. This is in contrast to the effective theories on the $AdS_3 \times S^3$ background in which the infinite Kaluza-Klein tower of massive spin-1 multiplets is realized as half maximal $\mathcal{N}=8$ supergravity theories [12] whose scalar target spaces are chosen from the series $\mathrm{SO}(8,n)/(\mathrm{SO}(8)\times\mathrm{SO}(n))$ and may thus accommodate an arbitrarily large number of degrees of freedom. The Kaluza-Klein spectrum (2.4) on the other hand contains already two copies of the multiplet $(\frac{1}{2},\frac{1}{2};\frac{1}{2},\frac{1}{2})_{\mathrm{S}}$ that together do not fit into the same maximal theory, not to mention the infinite number of higher massive multiplets. The effective theory describing the higher multiplets in the Kaluza-Klein spectrum on $AdS_3\times S^3\times S^3$ will presumably require an extension of the construction to a theory with an infinite number of supercharges of which all but a finite number become massive in a super Higgs effect at the ground state. The usual no-go theorem that excludes the existence of theories with more than 32 real supercharges relies on the assumption that the theory admits an unbroken phase, such that the spectrum around this symmetric ground state organizes into supermultiplets of the corresponding superalgebra. With the theory constructed in this paper we have found an example of a supersymmetric theory that has no symmetric phase but only a half supersymmetric ground state. Similarly, one could entertain the possibility of more than $\mathcal{N}=16$ supersymmetries in three dimensions which are necessarily broken at the ground state. Upon switching off the propagating matter, the theories should reduce to the $\mathcal{N}$-extended topological theory of [20]. We leave these questions for future study.

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A. $E_{8(8)}$ from various angles

The maximal supergravity theories in three dimensions are organized under the exceptional group $E_{8(8)}$. In particular, their scalar sector is given by a coset space σ model with target space $E_{8(8)}/\mathrm{SO}(16)$. In this appendix, we describe the Lie algebra $\mathfrak{e}_{8(8)}$ in different decompositions relevant for the embedding of the gauge group and construction of the embedding tensor in the main text.

A.1 $E_{8(8)}$ in the SO(16) basis

The 248-dimensional Lie algebra of $E_{8(8)}$ may be characterized starting from its 120-dimensional maximal compact subalgebra $\mathfrak{so}(16)$, spanned by generators $X^{IJ}=X^{[IJ]}$ with commutators

$$\left[X^{IJ},X^{KL}\right]=\delta^{JK}X^{IL}-\delta^{IK}X^{JL}-\delta^{JL}X^{IK}+\delta^{IL}X^{JK}\,, \tag{A.1}$$

where $I,J=1,\ldots,16$ denote SO(16) vector indices. The 128-dimensional non-compact part of $\mathfrak{e}_{8(8)}$ is spanned by generators Y^A which transform in the fundamental spinorial representation of SO(16), i.e. which satisfy commutators

$$[X^{IJ},Y^A]=-\frac{1}{2}\,\Gamma^{IJ}_{AB}\,Y^B\,,\qquad [Y^A,Y^B]=\frac{1}{4}\,\Gamma^{IJ}_{AB}\,X^{IJ}\,. \tag{A.2}$$

Here $A, B = 1, \ldots, 128$ label the spinor representation of SO(16) and $\Gamma^{IJ} = \Gamma^{[I}\Gamma^{J]}$ denotes the antisymmetrized product of SO(16) Γ-matrices. Moreover, we use indices $\dot{A}, \dot{B} = 1, \ldots, 128$ to label the conjugate spinor representation of SO(16). In the main text, indices $\mathcal{M}, \mathcal{N} = 1, \ldots, 248$ collectively label the full Lie algebra of $E_{8(8)}$, i.e. $\{t^{\mathcal{M}}\} = \{X^{IJ}, Y^A\}$ with

$$\left[t^{\mathcal{M}}, t^{\mathcal{N}}\right] = f^{\mathcal{M}\mathcal{N}}{}_{\mathcal{K}}\, t^{\mathcal{K}} . \tag{A.3}$$

The Cartan-Killing form finally is given by

$$\eta^{\mathcal{M}\mathcal{N}} = \frac{1}{60}\,\mathrm{tr}\,(t^{\mathcal{M}}\, t^{\mathcal{N}}) = \frac{1}{60}\, f^{\mathcal{M}\mathcal{K}}{}_{\mathcal{L}}\, f^{\mathcal{N}\mathcal{L}}{}_{\mathcal{K}} . \tag{A.4}$$

A.2 $E_{8(8)}$ in the SO(8, 8) basis

Alternatively, $\mathfrak{e}_{8(8)}$ may be built starting from its maximal subalgebra $\mathfrak{so}(8,8)$ spanned by 120 generators $X^{\mathtt{IJ}}$ with commutators

$$[X^{\mathtt{IJ}}, X^{\mathtt{KL}}] = \eta^{\mathtt{JK}} X^{\mathtt{IL}} - \eta^{\mathtt{IK}} X^{\mathtt{JL}} - \eta^{\mathtt{JL}} X^{\mathtt{IK}} + \eta^{\mathtt{IL}} X^{\mathtt{JK}} , \tag{A.5}$$

where $\mathtt{I}, \mathtt{J}, \ldots$ now denote vector indices of SO(8, 8) and $\eta_{\mathtt{IJ}} = \mathrm{diag}(-1, \ldots, -1, 1, \ldots, 1)$ is the SO(8, 8) invariant tensor. Similarly to the above, the full $\mathfrak{e}_{8(8)}$ is obtained by adding 128 generators $\hat{Q}_{\mathtt{A}}$, $\mathtt{A} = 1, \ldots, 128$, transforming in the spinor representation of SO(8, 8)

$$[X^{\mathtt{IJ}}, \hat{Q}_{\mathtt{A}}] = -\frac{1}{2}\Gamma^{\mathtt{IJ}}{}_{\mathtt{A}}{}^{\mathtt{B}}\, \hat{Q}_{\mathtt{B}} , \qquad [\hat{Q}_{\mathtt{A}}, \hat{Q}_{\mathtt{B}}] \;=\; \tfrac{1}{4}\, \eta_{\mathtt{IK}}\eta_{\mathtt{JL}}\eta_{\mathtt{BC}}\, \Gamma^{\mathtt{KL}}{}_{\mathtt{A}}{}^{\mathtt{C}}\, X^{\mathtt{IJ}} . \tag{A.6}$$

Here $\Gamma^{\mathtt{IJ}}{}_{\mathtt{A}}{}^{\mathtt{B}}$ denote the (rescaled) SO(8, 8)-generators in the spinor representation, i.e.

$$\Gamma^{\mathtt{IJ}}{}_{\mathtt{A}}{}^{\mathtt{B}} = \frac{1}{2}(\Gamma^{\mathtt{I}}{}_{\mathtt{A}}{}^{\dot{\mathtt{C}}}\,\overline{\Gamma}^{\mathtt{J}}{}_{\dot{\mathtt{C}}}{}^{\mathtt{B}} - \Gamma^{\mathtt{J}}{}_{\mathtt{A}}{}^{\dot{\mathtt{C}}}\,\overline{\Gamma}^{\mathtt{I}}{}_{\dot{\mathtt{C}}}{}^{\mathtt{B}}) , \tag{A.7}$$

where the gamma matrices satisfy

$$\Gamma^{\mathtt{I}}{}_{\mathtt{A}}{}^{\dot{\mathtt{C}}}\,\overline{\Gamma}^{\mathtt{J}}{}_{\dot{\mathtt{C}}}{}^{\mathtt{B}} + \Gamma^{\mathtt{J}}{}_{\mathtt{A}}{}^{\dot{\mathtt{C}}}\,\overline{\Gamma}^{\mathtt{I}}{}_{\dot{\mathtt{C}}}{}^{\mathtt{B}} = 2\eta^{\mathtt{IJ}}\, \delta_{\mathtt{A}}{}^{\mathtt{B}} , \tag{A.8}$$

with the transpose $\overline{\Gamma}$, and where $\mathtt{A}, \mathtt{B}, \ldots,$ denote spinor indices and $\dot{\mathtt{A}}, \dot{\mathtt{B}}, \ldots,$ conjugate spinor indices of SO(8, 8). It is important to note that in contrast to the SO(16) decomposition described above, spinor indices in these equations are raised and lowered not with the simple δ-symbol but with the corresponding SO(8, 8) invariant tensors $\eta_{\mathtt{AB}}$, $\eta_{\dot{\mathtt{A}}\dot{\mathtt{B}}}$ of indefinite signature (cf. (A.11) below).

A.3 $E_{8(8)}$ in the SO(8) $\times$ SO(8) basis

According to (3.13), (3.14) in the main text, the two decompositions of sections A.1 and A.2 may be translated into each other upon further breaking down to $\mathrm{SO}(8)_L \times \mathrm{SO}(8)_R$. To this end, we use the decomposition $\mathrm{SO}(8,8) \to \mathrm{SO}(8)_L \times \mathrm{SO}(8)_R$ with

$$\begin{aligned} &\mathbf{16}_V \to (\mathbf{8}_V, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{8}_V) , \\ &\mathbf{128}_S \to (\mathbf{8}_S, \mathbf{8}_C) \oplus (\mathbf{8}_C, \mathbf{8}_S) , \qquad \mathbf{128}_C \;\to\; (\mathbf{8}_S, \mathbf{8}_S) \oplus (\mathbf{8}_C, \mathbf{8}_C) , \end{aligned} \tag{A.9}$$

corresponding to the split of SO(8, 8) indices:

$$\mathtt{I} = (\hat{a}, b), \qquad \mathtt{A} = (\alpha\dot{\beta}\,, \dot{\gamma}\delta)\,, \qquad \dot{\mathtt{A}} = (\alpha\beta, \dot{\gamma}\dot{\delta})\,. \tag{A.10}$$

Here, $\hat{a}, \hat{b}, \ldots$ and $a, b, \ldots$ denote vector indices for the left and the right SO(8) factor, respectively, while $\alpha, \beta, \ldots$ and $\dot{\alpha}, \dot{\beta}, \ldots$ denote spinor and conjugate spinor indices, respectively, for both SO(8) factors. The invariant tensors $\eta^{\mathtt{IJ}}$, $\eta_{\mathtt{AB}}$ and $\eta_{\dot{\mathtt{A}}\dot{\mathtt{B}}}$ in this SO(8) notation take the form

$$\eta^{\mathtt{IJ}} = \begin{pmatrix} -\delta^{\hat{a}\hat{b}} & 0 \\ 0 & \delta^{ab} \end{pmatrix}, \qquad \eta_{\mathtt{AB}} = \begin{pmatrix} \eta_{\alpha\dot{\alpha},\beta\dot{\beta}} & 0 \\ 0 & \eta_{\dot{\alpha}\beta,\dot{\gamma}\delta} \end{pmatrix} = \begin{pmatrix} \delta_{\alpha\beta}\delta_{\dot{\alpha}\dot{\beta}} & 0 \\ 0 & -\delta_{\dot{\alpha}\dot{\gamma}}\delta_{\beta\delta} \end{pmatrix},$$
$$\eta_{\dot{\mathtt{A}}\dot{\mathtt{B}}} = \begin{pmatrix} \eta_{\alpha\beta,\gamma\delta} & 0 \\ 0 & \eta_{\dot{\alpha}\dot{\beta},\dot{\gamma}\dot{\delta}} \end{pmatrix} = \begin{pmatrix} \delta_{\alpha\gamma}\delta_{\beta\delta} & 0 \\ 0 & -\delta_{\dot{\alpha}\dot{\gamma}}\delta_{\dot{\beta}\dot{\delta}} \end{pmatrix}. \tag{A.11}$$

It is straightforward to verify, that the SO(8, 8) gamma matrices (A.8) can be expressed in terms of the SO(8) gamma matrices $\Gamma^a_{\alpha\dot{\gamma}}$ as (see also [26])

$$\begin{aligned} {\Gamma^a}_{\beta\dot{\gamma}}{}^{\delta\epsilon} &= \delta_{\beta\delta}\Gamma^a_{\epsilon\dot{\gamma}}\,, \qquad & {\Gamma^a}_{\dot{\alpha}\beta}{}^{\dot{\gamma}\dot{\delta}} &= -\delta_{\dot{\alpha}\dot{\gamma}}\Gamma^a_{\beta\dot{\delta}}\,, \\ {\Gamma^{\hat{a}}}_{\dot{\alpha}\beta}{}^{\gamma\delta} &= \delta_{\beta\delta}\Gamma^a_{\gamma\dot{\alpha}}\,, \qquad & {\Gamma^{\hat{a}}}_{\alpha\dot{\beta}}{}^{\dot{\gamma}\dot{\delta}} &= -\delta_{\dot{\beta}\dot{\delta}}\Gamma^a_{\alpha\dot{\gamma}}\,. \end{aligned} \tag{A.12}$$

With the results from the previous section, $\mathfrak{e}_{8(8)}$ can now explicitly be given in the $\mathfrak{so}(8)_L \oplus \mathfrak{so}(8)_R$ basis. Generators split according to $\{X^{ab}, X^{\hat{a}\hat{b}}, X^{a\hat{b}}, \hat{Q}_{\alpha\dot{\beta}}, \hat{Q}_{\dot{\gamma}\delta}\}$ with the commutation relations

$$\begin{aligned}
[X^{ab}, X^{cd}] &= \delta^{bc}X^{ad} - \delta^{ac}X^{bd} - \delta^{bd}X^{ac} + \delta^{ad}X^{bc}\,, \\
[X^{\hat{a}\hat{b}}, X^{\hat{c}\hat{d}}] &= -\delta^{\hat{b}\hat{c}}X^{\hat{a}\hat{d}} + \delta^{\hat{a}\hat{c}}X^{\hat{b}\hat{d}} + \delta^{\hat{b}\hat{d}}X^{\hat{a}\hat{c}} - \delta^{\hat{a}\hat{d}}X^{\hat{b}\hat{c}}\,, \\
[X^{ab}, X^{c\hat{d}}] &= \delta^{bc}X^{a\hat{d}} - \delta^{ac}X^{b\hat{d}}\,, \\
[X^{\hat{a}\hat{b}}, X^{c\hat{d}}] &= \delta^{\hat{a}\hat{d}}X^{c\hat{b}} - \delta^{\hat{b}\hat{d}}X^{c\hat{a}}\,, \\
[X^{a\hat{b}}, X^{c\hat{d}}] &= \delta^{\hat{b}\hat{d}}X^{ac} - \delta^{ac}X^{\hat{b}\hat{d}}\,, \\
[X^{ab}, \hat{Q}_{\alpha\dot{\beta}}] &= -\frac{1}{2}\overline{\Gamma}^{[a}_{\dot{\beta}\epsilon}\Gamma^{b]}_{\epsilon\dot{\delta}}\hat{Q}_{\alpha\dot{\delta}}\,, \qquad [X^{ab}, \hat{Q}_{\dot{\alpha}\beta}] \;=\; -\frac{1}{2}\Gamma^{[a}_{\beta\dot{\zeta}}\overline{\Gamma}^{b]}_{\dot{\zeta}\delta}\hat{Q}_{\dot{\alpha}\delta}\,, \\
[X^{\hat{a}\hat{b}}, \hat{Q}_{\alpha\dot{\beta}}] &= \frac{1}{2}\Gamma^{[a}_{\alpha\dot{\epsilon}}\overline{\Gamma}^{b]}_{\dot{\epsilon}\gamma}\hat{Q}_{\gamma\dot{\beta}}\,, \qquad [X^{\hat{a}\hat{b}}, \hat{Q}_{\dot{\alpha}\beta}] \;=\; \frac{1}{2}\overline{\Gamma}^{[a}_{\dot{\alpha}\epsilon}\Gamma^{b]}_{\epsilon\dot{\gamma}}\hat{Q}_{\dot{\gamma}\beta}\,, \\
[X^{a\hat{b}}, \hat{Q}_{\alpha\dot{\beta}}] &= \frac{1}{2}\Gamma^a_{\delta\dot{\beta}}\Gamma^b_{\alpha\dot{\gamma}}\hat{Q}_{\dot{\gamma}\delta}\,, \qquad [X^{a\hat{b}}\,, \hat{Q}_{\dot{\alpha}\beta}] \;=\; \frac{1}{2}\Gamma^a_{\beta\dot{\delta}}\Gamma^b_{\gamma\dot{\alpha}}\hat{Q}_{\gamma\dot{\delta}}\,, \\
[\hat{Q}_{\alpha\dot{\alpha}}, \hat{Q}_{\beta\dot{\beta}}] &= \frac{1}{4}\delta_{\alpha\beta}\overline{\Gamma}^{[a}_{\dot{\alpha}\gamma}\Gamma^{b]}_{\gamma\dot{\beta}}X^{ab} - \frac{1}{4}\delta_{\dot{\alpha}\dot{\beta}}\Gamma^{[a}_{\alpha\dot{\gamma}}\overline{\Gamma}^{b]}_{\dot{\gamma}\beta}X^{\hat{a}\hat{b}}\,, \\
[\hat{Q}_{\alpha\dot{\alpha}}, \hat{Q}_{\dot{\beta}\beta}] &= -\frac{1}{2}\Gamma^a_{\beta\dot{\alpha}}\Gamma^b_{\alpha\dot{\beta}}X^{a\hat{b}}\,, \\
[\hat{Q}_{\dot{\alpha}\alpha}, \hat{Q}_{\dot{\beta}\beta}] &= \frac{1}{4}\delta_{\alpha\beta}\overline{\Gamma}^{[a}_{\dot{\alpha}\gamma}\Gamma^{b]}_{\gamma\dot{\beta}}X^{\hat{a}\hat{b}} - \frac{1}{4}\delta_{\dot{\alpha}\dot{\beta}}\Gamma^{[a}_{\alpha\dot{\gamma}}\overline{\Gamma}^{b]}_{\dot{\gamma}\beta}X^{ab}\,.
\end{aligned} \tag{A.13}$$

Moreover, the Cartan-Killing form (A.4) can be computed in the $\mathrm{SO}(8)\times\mathrm{SO}(8)$ basis by use of this explicit form of the structure constants. The result is

$$
\begin{aligned}
&\eta^{ab,cd} = -\delta^{[ab],[cd]}\,, \qquad \eta^{\hat a\hat b,\hat c\hat d} = -\delta^{[\hat a\hat b],[\hat c\hat d]}\,, \qquad \eta^{a\hat b,c\hat d} = \delta^{ac}\delta^{\hat b\hat d}\,,\\
&\eta_{\alpha\dot\beta,\gamma\dot\delta} = \delta_{\alpha\gamma}\delta_{\dot\beta\dot\delta}\,, \qquad \eta_{\dot\alpha\beta,\dot\gamma\delta} = -\delta_{\dot\alpha\dot\gamma}\delta_{\beta\delta}\,,
\end{aligned}
\tag{A.14}
$$

while all other components vanish.

Finally let us identify explicitly the $\mathrm{SO}(16)$ subalgebra in this $\mathrm{SO}(8,8)$ basis. With respect to the $\mathrm{SO}(8)\times\mathrm{SO}(8)$ decomposition of $\mathrm{SO}(16)$ in (2.15) the indices split according to

$$
I = (\dot\alpha, \beta)\,, \qquad A = (\alpha\dot\beta, \hat a b)\,, \qquad \dot A = (\alpha a, \hat b\dot\beta)\,. \tag{A.15}
$$

Correspondingly, the $\mathrm{SO}(16)$ generators X^{IJ} decompose into $X^{\alpha\beta}$, $X^{\dot\alpha\dot\beta}$ and $X^{\alpha\dot\beta}$, and can be written in terms of the compact $E_{8(8)}$ generators by

$$
X^{\alpha\beta} = \frac{1}{2}\Gamma^{ab}_{\alpha\beta}X^{ab}\,, \qquad X^{\dot\alpha\dot\beta} = -\frac{1}{2}\Gamma^{\hat a\hat b}_{\dot\alpha\dot\beta}X^{\hat a\hat b}\,, \qquad X^{\alpha\dot\beta} = \hat Q_{\dot\beta\alpha}\,. \tag{A.16}
$$

That these satisfy the $\mathrm{SO}(16)$ algebra can be verified explicitly by use of standard gamma matrix identities. The noncompact generators Y^A are identified as

$$
Y^{\alpha\dot\beta} = \hat Q_{\alpha\dot\beta}\,, \qquad Y^{\hat a b} = X^{b\hat a}\,. \tag{A.17}
$$

One immediately verifies that this split into compact and noncompact generators is in agreement with the eigenvalues of the Cartan-Killing form (A.14).

A.4 $E_{8(8)}$ in the $\mathrm{SO}(4)\times\mathrm{SO}(4)$ basis

To explicitly describe the embedding of the gauge group $G_0 = G_\mathrm{c} \ltimes (\hat T_{34}, T_{12})$ described in section 3.2, we finally need the decomposition of $E_{8(8)}$ under the $\mathrm{SO}(4)_L\times\mathrm{SO}(4)_R$ from (2.2). This is obtained from the previous section upon further decomposition according to (2.13), (2.14). In $\mathrm{SO}(8)_R$ indices a, α, $\dot\alpha$, this corresponds to the splits

$$
a = ([ij], 0, \bar 0)\,, \qquad \alpha = (i,j)\,, \qquad \dot\alpha = (i,j)\,, \tag{A.18}
$$

and similarly for $\mathrm{SO}(8)_L$. Here, $i,j,\ldots$ denote $\mathrm{SO}(4)$ vector indices. The $\mathrm{SO}(8)$ gamma matrices can then be expressed in terms of the invariant $\mathrm{SO}(4)$ tensors δ^{ij} and ε^{ijkl} as

$$
\Gamma^{ij} = \begin{pmatrix} \varepsilon^{ij} & 2\delta^{ij} \\ -2\delta^{ij} & \varepsilon^{ij} \end{pmatrix}, \qquad \Gamma^{0} = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}, \qquad \Gamma^{\bar 0} = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}, \tag{A.19}
$$

with the 4×4 matrices

$$
\mathbf{1}_{kl} = \delta_{kl}\,, \qquad (\varepsilon^{ij})_{kl} = \varepsilon^{ijkl}\,, \qquad (\delta^{ij})_{kl} = \delta^{ij}_{kl} = \delta^{i[k}\delta^{l]j}\,. \tag{A.20}
$$

It is straightforward to check that the matrices (A.19) satisfy the standard Clifford algebra, making use of the relations

$$
\delta^{ij}(\delta^{mn})^t + \delta^{mn}(\delta^{ij})^t + \varepsilon^{ij}(\varepsilon^{mn})^t + \varepsilon^{mn}(\varepsilon^{ij})^t = 2\delta^{ij,mn}\,\mathbf{1}\,, \tag{A.21}
$$

$$\varepsilon^{ij}(\delta^{mn})^t + \varepsilon^{mn}(\delta^{ij})^t - \delta^{ij}(\varepsilon^{mn})^t - \delta^{mn}(\varepsilon^{ij})^t = 0\,, \tag{A.22}$$

which can be proved using the identity $\varepsilon^{[ijkl}\delta_n^{m]} = 0$. Next we have to decompose these Γ-matrices into selfdual and anti-selfdual parts, corresponding to (A.18),

$$\Gamma_\pm^{ij} = \frac{1}{\sqrt{2}}\left(\Gamma^{ij} \pm \frac{1}{2}\varepsilon^{ijkl}\Gamma^{kl}\right), \tag{A.23}$$

such that $\tilde{\Gamma}_\pm^{ij} := \frac{1}{2}\varepsilon^{ijkl}\Gamma_\pm^{kl} = \pm\Gamma_\pm^{ij}$. Inserting the representation (A.19) of Γ-matrices into the structure constants in (A.13) eventually yields the decomposition of $\mathfrak{e}_{8(8)}$ in the $\mathfrak{so}(4)_L \oplus \mathfrak{so}(4)_R$ basis.

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